By universal coefficients, $H^n(X; G) \cong \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G)$, so cohomology is not really anything "new"; the groups themselves provide no new information to distinguish spaces. So why should we care? And what does it measure, anyway? One way to answer that question is to study what a cochain is, and what it means for a cocycle to not be a coboundary.

Let us think in terms of simplicial cochains for a Δ-cplx X. An n-cochain assigns elements of G to the n-simplices of X. Think for example of a 0-cochain φ, which assigns values to the vertices. The coboundary $\delta\varphi$ assigns the difference $\varphi(v_1) - \varphi(v_0)$ to each oriented 1-simplex $[v_0, v_1]$. A 1-cochain, assigning values to each 1-simplex, is a coboundary if those values represent the *differences* of a fcn defined on the vertices. A 1-cocycle, on the other hand, is a 1-cochain ψ for which

$$\delta\psi([v_0, v_1, v_2]) = \psi([v_0, v_1]) - \psi([v_0, v_2]) + \psi([v_0, v_1]) = 0 \text{ (*)}$$

for every 2-simplex (since the map is $0 \Leftrightarrow$ it is 0 on each basis element). The point is that if a 1-cochain represents the differences of the values of some globally defined function on the vertices, across each 1-simplex, then (*) must certainly be true; $(a - b) + (b - c) + (c - a) = 0$. This is what 1-coboundary $\Rightarrow$ 1-cocycle says. The fact that the opposite need not be true is a reflection of the topology of X; the condition (*) essentially says that the values on edges represent differences of values on the vertices "locally". This can be related to the idea of vector fields versus conservative vector fields, in analysis; a vector field will integrate around the boundary of a 2-simplex to give 0, but a vector field is conservative (equal to the gradient of a function) $\Leftrightarrow$ it integrates to 0 around every closed loop. In higher degrees we can view things analogously, and these can be related, analytically, to integrals over correspondingly higher dimensional regions.

As mentioned previously, essentially all of the machinery we built to study homology can be adapted to study cohomology. A map $f : X \to Y$ induces a (chain) map

$$f_\# : C_n(X) \to C_n(Y);$$

dualizing, we get a (chain) map

$$f^\# = f_\#^* : C_n^*(Y; G) \to C_n^*(X; G),$$

which gives a homomorphism

$$f^* : H^n(Y; G) \to H^n(X; G).$$

This satisfies $(f \circ g)^* = g^* \circ f^*$ (since this is true for the chain complexes and carries over to the duals) and $I^* = I$, so we immediately recover the analogous result that a homeo induces isos on cohomology. More, homotopic maps induce equal maps on cohomology, since if $f, g : X \to Y$ are homotopic, then $f_\#, g_\#$ are chain homotopic, via a chain homotopy

$$H;\ H\partial + \partial H = f_\# - g_\#.$$

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Dualizing, we obtain a "chain cohomotopy" (?) H^*, <u>decreasing</u> degree by one, with

$$\delta H^* + H^* \delta = f^\# - g^\#$$

and therefore, by the same proof, $f^\#$ and $g^\#$ induce the same map $f^* = g^*$ on cohomology. Consequently, we recover the result that a homotopy equivalence between spaces induces isomorphisms between their cohomology groups.

We can define relative cohomology groups $H^n(X,A;G)$ by defining
$$C^n(X,A;G) = \mathrm{Hom}(C_n(X,A),G) = \mathrm{Hom}(C_n(X)/C_n(A),G).$$
Maps of pairs induce, as in the homology case, homomorphisms between relative cohomology groups. Dualizing the SES
$$0 \to C_n(A) \xrightarrow{\iota} C_n(X) \xrightarrow{p} C_n(X)/C_n(A) \to 0$$
we get a sequence
$$(*)\ 0 \to \mathrm{Hom}(C_n(X)/C_n(A),G) \xrightarrow{p^*} \mathrm{Hom}(C_n(X),G) \xrightarrow{\iota^*} \mathrm{Hom}(C_n(A),G) \to 0$$
which turns out to be exact; this is basically because $C_n(X)/C_n(A)$ has basis chains in X that do not map completely into A, so really
$$C_n(X) = C_n(A) \oplus C_n(X)/C_n(A),$$
which consequently means that
$$\mathrm{Hom}(C_n(X),G) \cong \mathrm{Hom}(C_n(A),G) \oplus \mathrm{Hom}(C^n(X)/C_n(A),G),$$
and under the isomorphism the sequence (*) becomes the "obvious" one, which is exact. The coboundary maps
$$\delta : \mathrm{Hom}(C_n(X)/C_n(A),G) \to \mathrm{Hom}(C_{n+1}(X)/C_{n+1}(A),G)$$
are given by, for $\varphi : C_n(X)/C_n(A) \to G$, composing with p to get a map $\varphi_1 = \varphi \circ p : C_n(X) \to G$, taking its coboundary (in X), and noting that the resulting map $\delta\varphi_1 = \psi_1$ is 0 on $C_{n+1}(A)$ (since $\psi_1(a) = \varphi_1(\partial a) = \varphi(p(\partial a)) = \varphi(0) = 0$, since $\partial a \in C_n(A)$, so $p(\partial a) = 0$.

These exact sequences (*) consequently, as before, give rise to a LEHS (LECS?)

$$\cdots \to H^{n+1}(A;G) \to H^n(X,A;G) \to H^n(X;G) \to H^n(A;G) \to H^{n-1}(X,A;G) \to \cdots$$

Similarly there is a LEHS for a triple $B \subseteq A \subseteq X$. The universal coefficients theorem applies to relative homology, since the relevant chain groups are free abelian, so

$$H^n(X,A;G) \cong \mathrm{Hom}(H_n(X,A),G) \oplus \mathrm{Ext}(H_{n-1}(X,A),G)$$

Cohomology on small chains can be defined analogously; $H^n_{\mathcal{U}}(X;G)$ is the homology of the cochain complex $\mathrm{Hom}(C_n^{\mathcal{U}}(X),G)$. The inclusion-induced map

$$\iota^{\#} : \mathrm{Hom}(C_n(X),G) \to \mathrm{Hom}(C_n^{\mathcal{U}}(X),G)$$

induces isomorphisms of cohomology groups, by dualizing the proof we <u>didn't</u> do for homology, namely that there is a chain map $b : C_n(X) \to C_n^{\mathcal{U}}(X)$ such that $\iota \circ b$ and $b \circ \iota$ are chain homotopic to the identity. As above, the duals of the chain homotopies form the necessary chain (co)homotopies. We can then recover excision:

If $A, B \subseteq$ satisfy the usual requirements for excsion, then
$\iota^* : H^n(X,A;G) \to H^n(B, A\cap B;G)$ is an isomorphism.

the proof of building the iso

$$H^n(C_n(X)/C_n(A);G) \to H_n(\mathrm{Hom}(C_n^{\{A,B\}}(X)/C_n(A),G))$$

via SESs (arguing as above that the duals of the relevant SESs are exact) and the Five Lemma applied to the resulting LEHSs, together with

$$\mathrm{Hom}(C_n^{\{A,B\}}(X)/C_n(A),G) \cong \mathrm{Hom}(C_n(B)/C_n(A\cap B),G)$$

because the domains are isomorphic, inducing the corresponding iso in cohomology, goes through without change.

The same reinterpretation from homology also gives the excision isomorphism

$$H^n(X, A; G) \to H^n(X \setminus B, A \setminus B; G).$$

Dualizing the SES

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n^{\{A,B\}}(X) \to 0$$

gives the (short exact, by the argument above) sequence

$$0 \to \mathrm{Hom}(C_n^{\{A,B\}}(X), G) \to \mathrm{Hom}(C_n(A), G) \oplus \mathrm{Hom}(C_n(B), G) \to \mathrm{Hom}(C_n(A \cap B), G) \to 0$$

yielding the Mayer-Vietoris sequence for cohomology:

$$\cdots \to H^{n-1}(A \cap B) \to H^n(X) \to H^n(A) \oplus H^n(B) \to H^n(A \cap B) \to H^{n+1}(X) \to \cdots$$

(supressing the coefficient group G to make this fit on a line...).

Reduced cohomology can be defined by taking the dual of the augmented chain complex defining homology. As with homology, $\widetilde{H}^n(X; G) \cong H^n(X; G)$ for $n \geq 1$ and, from the universal coefficients theorem, $\widetilde{H}^0(X; G) \cong \mathrm{Hom}(\widetilde{H}_0(X), G)$. We can think of $\mathrm{Hom}(\widetilde{H}_0(X), G)$ as a direct product of Gs, one for each path component of X; alternatively, this is the set of all functions from X to G that are constant on path components. $\mathrm{Hom}(\widetilde{H}_0(X), G)$ is slightly smaller; since the augmentation map $C_0(X) \to \mathbb{Z}$ sends every point to 1, the dual map

$$G \cong \mathrm{Hom}(\mathbb{Z}, G) \to \mathrm{Hom}(C_0(X), G) \to \cdots$$

sends $g \in G$ to the map which sends each basis element of $C_0(X)$ (i.e., points) to g. So $\widetilde{H}^0(X; G)$ is built by modding outm in addition, by this map; so $\widetilde{H}^0(X; G)$ can be identified with the set of *non-constant* functions from the path-components of X to G.

As for homology, a combination of excision, the LES of a pair, and the Five Lemma implies that $H^n(X, A; G) \cong \widetilde{H}^n(X/A; G)$ when A has a neighborhood that deformation retracts to it.

With these facts in hand, we can carry out calculations of cohomology groups in much the same spirit as we did for homology. For example, Mayer-Vietoris and induction implies that $H^k(S^n; G) \cong G$ for $k = 0, n$ and 0 otherwise. In most cases, though, if all that we are after are the groups themselves, the universal coefficients theorem provides a faster computational route. In fact, at least for $\mathbb{Z}$-coefficients, if we define the torsion subgroup T of an abelian group G to be the set of elements of finite order in G, then if the homology groups of X are finitely generated, then $\mathrm{Hom}(H_n(X), \mathbb{Z})$ is isomorphic to the free abelian part of $H_n(X)$ (which is isomorphic to H_n/T_n), and $\mathrm{Ext}(H_{n-1}(X), \mathbb{Z})$ is the torsion part T_{n-1} of $H_{n-1}(X)$, so

$$H^n(X; \mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$$

is the direct sum of the free part of $H_n(X)$ and the torsion part of $H_{n-1}(X)$, so the difference between homology and cohomology is that cohomology carries its torsion one degree higher than homology does! With more general coefficients, the situation becomes more involved...