**Homology with coefficients:** The chain complexes that we have dealt with so far have had elements which are $\mathbb{Z}$-linear combinations of basis elements (which are themselves singular simplices or equivalence classes of them); that is, the chain groups have been free abelian groups. Formally, though, there is no reason to restrict ourselves to $\mathbb{Z}$ coefficients; all that is required is to add coefficients and take the negative of a chain (i.e., the negatives of its coefficients). Any abelian group $G$ can be used as coefficient group; the (singular or simplicial) chain complex $C_*(X;G)$ consists of the groups $C_n(X;G) = \{\sum g_i\sigma_i : \sigma_i : \Delta^i \to X\}$ of formal $G$-linear combinations of (singular or actual) $i$-simplices in $X$. Defining boundary maps as before, we write $\partial_i g\sigma = \sum(-1)^j g\sigma|_{\text{faces}}$; $(-1)^j$ isn't thought of as an integer, but as altering the coefficient assigned to a face, between $g$ and $-g$. Our formulas then go through unchanged, to show that $\partial^2 = 0$, so $C_*(X;G)$ is a chain complex. Its homology groups are called the (simplicial or singular) *homology groups of $X$ with coefficients in $G$*, denoted $H_*(X;G)$. "Ordinary" homology, in this context, would be denoted $H_*(X;\mathbb{Z})$

The machinery we have built up to work with ordinary homology carries over to homology with coefficients; none of our proofs really used the fact that our coefficient group was $\mathbb{Z}$. With one exception; our computations of simplicial homology groups, via Smith normal form, specifically used row and column operations over $\mathbb{Z}$. But this points out the fact that it is really linear algebra that is required to carry out computations; if we choose a coefficient group which is (the additive group of) a <u>field</u> $F$, then we can treat our chain groups $C_n(X;F)$ as vector spaces over $F$, and the boundary maps are linear transformations, where we know that $B_i = \text{im } \partial_{i+1}$ is a subvector space of $Z_i = \ker \partial_i$, and $H_i(X;F) = Z_i/B_i$ is a vector space over $F$, i.e., $H_i(X;F) \cong F^{n_i}$ for some $n_i$ (possibly infinite).

[We use the field structure to <u>prove</u> this, but never need the multiplicative structure to present it. There <u>is</u> something to prove here, though: if $z$ is a cycle, so is $az$ for any $a \in F$ (formally, this is multiplying coeffiecients), and the same is true for boundaries. But the point is that we are imposing the vector space structure from the "outside" (the computations don't care), and noting that from this point of view $\partial(az) = a\partial z$, i.e., that $\partial$ is a linear transformation. This <u>helps</u> us do computations, but isn't <u>required</u> to do them.] Popular coefficient groups to use are $\mathbb{Z}_n, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$.

We can also introduce reduced homology with coefficients, by augmenting a chain complex with an evaluation map $C_0(X;G) \to G \to 0$, which takes the sum of the coefficients of a 0-chain. We can also extend these ideas to cellular homology. Working through the computations again, we find, for a CW-complex $X$, that

$$H_n(X^{(n)}, X^{(n-1)}; G) \cong \widetilde{H}_n(\vee S^n; G) \cong \oplus G,$$

one summand for each $n$-cell, and we can construct cellular homology with coefficients and show that $H_n^{CW}(X;G) \cong H_n(X;G)$ in the exact same way (the same of course, is true of simplicial homology with coefficients). The only point to really make is that the computation of the cellular boundary maps

$$H_n(X^{(n)}, X^{(n-1)}; G) \to H_{n-1}(X^{(n-1)}, X^{(n-2)}; G)$$

again amounts to computing the maps

$$f_* : G = H_{n-1}(S^{n-1}; G) \to H_{n-1}(S^{n-1}; G) = G$$

induced by the attaching map

$$f : S^{n-1} \to X^{(n-1)} \to X^{(n-1)}/X^{(n-2)} \cong \vee S^{n-1} \to S^{n-1}$$

of an $n$-cell; if this map has degree $m$, then $f_* : G \to G$ is multiplication by $m$.

So, for example, we can compute the homology groups $H_k(\mathbb{R}P^n;\mathbb{Z}_2)$ via cellular homology by noting that using the standard CW-structure with one cell in each dimension, where before the cellular boundary maps $\mathbb{Z}\to\mathbb{Z}$ were either 0 or multiplication by 2, now the maps $\mathbb{Z}_2\to\mathbb{Z}_2$ are all 0 (since in $\mathbb{Z}_2$ multiplication by 2 is 0). So in every computation ker $=\mathbb{Z}_2$ and im $=0$, so $H_k(\mathbb{R}P^n;\mathbb{Z}_2)\cong\mathbb{Z}_2$ for all $0\le k\le n$, and is 0 otherwise.

Coefficients do introduce one additional feature; a homomorphism of groups $\varphi : G\to H$ induces chain maps on complexes $C_n(X;G)\to C_n(X;H)$, via $\sum g_i\sigma_i\mapsto\sum\varphi(g_i)\sigma_i$. Even more, a short exact sequence of coefficient groups $0\to K\to G\to H\to 0$ induces SESs of chain complexes $0\to C_*(X;K)\to C_*(X;G)\to C_*(X;H)\to 0$, giving LEHSs interweaving the homology groups of $X$ with coefficients in $G,H$, and $K$. E.g., the SES

$$0\to\mathbb{Z}\stackrel{\times m}{\to}\mathbb{Z}\to\mathbb{Z}_m\to 0$$

gives the LEHS

$$\cdots\to H_n(X)\stackrel{(\times m)_*}{\to}H_n(X)\to H_n(X;\mathbb{Z}_m)\to H_{n-1}(X)\stackrel{(\times m)_*}{\to}H_{n-1}(X)\to\cdots.$$

Using the meta-fact that an exact sequence

$$A\stackrel{\alpha}{\to}B\stackrel{\beta}{\to}C\stackrel{\gamma}{\to}D\stackrel{\delta}{\to}E$$

yields the short exact sequence

$$0\to B/\text{im }\alpha\stackrel{\overline{\beta}}{\to}C\stackrel{\gamma}{\to}\ker\delta\to 0$$

(since im $\alpha=\ker\beta$ and im $\gamma=\ker\delta$), we get

$$0 \to H_n(X)/\{m[z] : [z] \in H_n(X)\} \to H_n(X;\mathbb{Z}_m) \to \{[z] \in H_{n-1}(X) : m[z] = 0\} \to 0$$

is exact. If $m = p =$ prime, then knowing the homology of $X$ allows us to compute $H_n(X;\mathbb{Z}_p)$, since we can compute the two groups on the ends (hence their dimensions over $\mathbb{Z}_p$), which allows us to compute the dimension of $H_n(X;\mathbb{Z}_p)$, hence compute $H_n(X;\mathbb{Z}_p)$. This last part follows since the alternating sum of the dimensions over $\mathbb{Z}_p$ of the terms in this exact sequence is 0, for the exact same reason this is true about ranks over $\mathbb{Z}$: such a number is invariant under taking homology (the proof for rank over $\mathbb{Z}$ goes through unchanged).

Homology with coefficients can, in some instances, tell us things that ordinary homology can't. For example, $X = \mathbb{R}P^2$ has reduced homology $\mathbb{Z}_2$ in dimension 1 only, and so the quotient map $q : X \to X/X^{(1)} \cong S^2$ induces the trivial map on all homology groups, using $\mathbb{Z}$ coefficients. But using $\mathbb{Z}_2$-coefficients, the LES of the pair $(X^{(1)}, X)$ gives, in part,

$$\cdots \to 0 = \widetilde{H}_2(X^{(1)};\mathbb{Z}_2) \to \widetilde{H}_2(X;\mathbb{Z}_2) \xrightarrow{q_*} \widetilde{H}_2(X/X^{(1)};\mathbb{Z}_2) \to \cdots$$

so $q_* : \mathbb{Z}_2 = \widetilde{H}_2(X;\mathbb{Z}_2) \to \widetilde{H}_2(X/X^{(1)};\mathbb{Z}_2)$ is injective, hence non-trivial, so $q$ is not a homotopically trivial map. Which is something that could not be concluded from the induced map on ordinary homology.

As another example, homology with $\mathbb{Z}_2$-coefficients play a role in a proof of the **Borsuk-Ulam Theorem**: For every map $f : S^n \to \mathbb{R}^n$, there is an $x \in S^n$ with $f(x) = f(-x)$. As part of our proof, we need: if $f : S^n \to S^n$ is an odd map (i.e., $f(-x) = -f(x)$), then $f$ has odd degree. The idea is that $f$ induces a map $g : \mathbb{R}P^n \to \mathbb{R}P^n$ (first take the composition $S^n \to S^n \to \mathbb{R}P^n$, and note that it factors through a map from $\mathbb{R}P^n$) satisfying $g \circ q = q \circ f$.

$q : S^n \to \mathbb{R}P^n$ is a (2-sheeted) covering space map. There is a short exact sequence of chain complexes

$$0 \to C_n(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\tau} C_n(S^n; \mathbb{Z}_2) \xrightarrow{q_\#} C_n(\mathbb{R}P^n; \mathbb{Z}_2) \to 0$$

where $\tau$ is the *transfer map* $\tau(\sigma) = \widetilde{\sigma}_1 + \widetilde{\sigma}_2$, the sum of the two lifts of a singular simplex into $\mathbb{R}P^n$, to $S^n$; the lifts exist by the lifting criterion, since the domain, $\Delta^n$, of $\sigma$ is contractible. (This last statement also shows that $q_\#$ is surjective.) $q_\#(\widetilde{\sigma}_1 + \widetilde{\sigma}_2) = \sigma + \sigma = 0$, since we are using $\mathbb{Z}_2$-coefficients; OTOH, if , $q_\#(\sum \sigma_i) = 0$, then the $\sigma_i$ must occur as pairs of lifts of singular simplices in $\mathbb{R}P^n$, giving exactness at the middle term. Finally, the transfer map is injective, since $\tau(\sum \sigma_i) = 0$ means that the $\widetilde{\sigma}_{i,j}$ must pair off. They can't do this as $\widetilde{\sigma}_{i,1}, \widetilde{\sigma}_{i,2}$ (they don't agree as maps, since they send the basepoint to different points), so they must pair with different initial indices. Then $\sigma_{i_1} = q \circ \widetilde{\sigma}_{i_1,j_1} = q \circ \widetilde{\sigma}_{i_2,j_2} = \sigma_{i_2}$ means that we can eliminate $\sigma_{i_1} + \sigma_{i_2} = 0$ from the sum; finitely many repetitions give $\sum \sigma_i = 0$.

We therefore get a long exact homology sequence

$$\cdots 0 \to H_n(\mathbb{R}P^n;\mathbb{Z}_2) \xrightarrow{\tau_*} H_n(S^n;\mathbb{Z}_2) \xrightarrow{q_*} H_n(\mathbb{R}P^n;\mathbb{Z}_2) \xrightarrow{\partial} H_{n-1}(\mathbb{R}P^n;\mathbb{Z}_2) \to 0 \cdots$$
$$\cdots 0 \to H_i(\mathbb{R}P^n;\mathbb{Z}_2) \xrightarrow{\partial} H_{i-1}(\mathbb{R}P^n;\mathbb{Z}_2) \to 0 \cdots$$
$$\cdots 0 \to H_1(\mathbb{R}P^n;\mathbb{Z}_2) \xrightarrow{\partial} H_0(\mathbb{R}P^n;\mathbb{Z}_2) \xrightarrow{\tau_*} H_0(S^n;\mathbb{Z}_2) \xrightarrow{q_*} H_0(\mathbb{R}P^n;\mathbb{Z}_2) \to 0$$

(the two 0's at top are $\widetilde{H}_{n+1}(\mathbb{R}P^n;\mathbb{Z}_2)$ and $\widetilde{H}_{n-1}(S^n;\mathbb{Z}_2)$, the rest are similar) which, plugging in the known values for the groups is

$$0 \to \mathbb{Z}_2 \xrightarrow{\tau_*} \mathbb{Z}_2 \xrightarrow{q_*} \mathbb{Z}_2 \xrightarrow{\partial} \mathbb{Z}_2 \to 0 \cdots 0 \to \mathbb{Z}_2 \xrightarrow{\partial} \mathbb{Z}_2 \to 0 \cdots 0 \to \mathbb{Z}_2 \xrightarrow{\partial} \mathbb{Z}_2 \xrightarrow{\tau_*} \mathbb{Z}_2 \xrightarrow{q_*} \mathbb{Z}_2 \to 0$$

But for the initial part, $\tau_*$ is then injective, hence surjective, and $\partial$ is surjective, hence injective, so $q_*$ is the zero map. In the middle part we have isomorphisms, while the final part again is a zero map between two isomorphisms.

As we have remarked, an odd map $f : S^n \to S^n$ induces a map $g : \mathbb{R}P^n \to \mathbb{R}P^n$, which in turn induce maps between chain groups in two short exact transfer sequences. These maps are chain maps; by definition they commute with the induced map on chains from the quotient map, and they commute with the transfer map since the two lifts of $g \circ \sigma$ are $f \circ \widetilde{\sigma}_i$, since these <u>are</u> lifts and lifts are unique. So their sums are the same, implying that $\tau \circ g = f \circ \tau$.

These therefore descend to maps between the corresponding two long exact transfer sequences. This, it turns out, allows us to pull ourselves up by our bootstraps, allowing us to infer information about the map $g_* : H_n(\mathbb{R}P^n;\mathbb{Z}_2) \to H_n(\mathbb{R}P^n;\mathbb{Z}_2)$ from information about the map $g_* : H_0(\mathbb{R}P^n;\mathbb{Z}_2) \to H_0(\mathbb{R}P^n;\mathbb{Z}_2)$ .

In the commutative squares

$$\begin{array}{ccc} H_i(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\partial} & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}_2) \\ {\scriptstyle g_*}\downarrow & & {\scriptstyle g_*}\downarrow \\ H_i(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\partial} & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}_2) \end{array} \quad , 1 \leq i \leq n, \quad \begin{array}{ccc} H_n(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\tau_*} & H_n(S^n;\mathbb{Z}_2) \\ {\scriptstyle g_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ H_n(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\tau_*} & H_n(S^n;\mathbb{Z}_2) \end{array}$$

the horizontal maps are isomorphisms, and for the first, the vertical arrow on the right is an isomorphism (by induction; the base case is $H_0$ which amounts to saying that $g$ induces a bijection on path components), so the arrow on the left is an isomorphism. For the second, the vertical arrow on the left is an isomorphism by the argument just given, so the arrow on the right is an isomorphism. But this arrow is the induced map on top-dimensional homology of an $n$-sphere, and so by our discussion on cellular homology is multiplication, in $\mathbb{Z}_2$, by the degree of the map $f$. If this degree were even, then the map on $\mathbb{Z}_2$ would be the 0 map; therefore, the degree is odd. Which finishes the proof of the preliminary result.

For the proof of Borsuk-Ulam, given $f : S^n \to \mathbb{R}^n$, suppose that $f(x) \neq f(-x)$ for every $x$. Then the fcn $g(x) = f(x) - f(-x)$ never takes the value 0, so the fcn $h : S^n \to S^{n-1}$ given by $h(x) = g(x)/||g(x)||$ is cts. But $h(-x) = -h(x)$, so the fcn $k = h|S^{n-1} : S^{n-1} \to S^{n-1}$ is an odd fcn, and so has odd degree, and, in particular, induces a non-trivial map on the level of $H_{n-1}(S^{n-1})$. But this map factors through $H_{n-1}(S^n) = 0$, since $k = h \circ \iota : S^{n-1} \to S^n \to S^{n-1}$, a contradiction. So there must be an $x$ with $f(x) = f(-x)$. So, e.g, there are somewhere on Earth antipodal points that have both the exact same level of background radiation and the same annual rainfall (or any two other contiouously varying quantities you care to name).