Connecting homomorphisms: There is a reason the connecting homomorphism in the LEHS is denoted by "∂": usually that's what it is. For example, in the LES of a pair,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

starting with a relative cycle $[z] \in C_n(X)/C_n(A)$, $\partial[z] = 0$, this means that $z \in C_n(X)$ and $\partial z \in C_{n-1}(A)$, and the connecting homomorphism simply chooses $a = \partial z$, since $\partial a = \partial\partial z = 0$, so $a \in Z_{n-1}(A)$. Similarly, in the Mayer-Vietoris sequence,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_n(A \cap B) & \longrightarrow & C_n(A) \oplus C_n(B) & \longrightarrow & C_n^{\{A,B\}}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A \cap B) & \longrightarrow & C_{n-1}(A) \oplus C_{n-1}(B) & \longrightarrow & C_{n-1}^{\{A,B\}}(X) & \longrightarrow & 0
\end{array}$$

a cycle $z \in C_n^{\{A,B\}}(X)$, $\partial z = 0$, can be expressed as $z = a + b$ with $a \in C_n(A), b \in C_n(B)$, so $\partial b = -\partial a$ is in both $C_{n-1}(B)$ and $C_{n-1}(A)$, so $w = \partial a \in C_{n-1}(A \cap B)$. Which is what the connecting homomorphism chooses, since $\partial w = \partial\partial a = 0$, so $w \in Z_{n-1}(A \cap B)$.

So, for example, the isomorphism $\partial : H_n(S^n) \to H_{n-1}(S^{n-1})$ amounts to <u>choosing</u> a generator w for $H_{n-1}(S^{n-1})$ (inductively, essentially, a homeomorhism $h : \partial\Delta^n \to S^{n-1}$ treated as a sum of singular simplices) and choosing chains w_+, w_- in S^n_+ and S^n_- with boundaries equal to h (essentially, the obvious extensions of h mapping Δ^n onto $S^n_\pm$); then $w_+ - w_-$ is a cycle in in $C_n(S^n)$ and maps onto w under ∂. We leave it to the interested observer to verify our inductive claim that $w_+ - w_-$ is "really" a homeomorphism $\partial\Delta^{n+1} \to S^n$ represented as a sum of singular simplices.