In the end, the big result that allows us to get our homology machine really running is what is known as **excision**. In a sense, it is the analogue of Seifert - van Kampen. We start with $X = A \cup B$, and we want to express the homology of X in terms of that of A, B, and $A \cap B$. With long exact homology sequences in mind, we try to first build a short exact sequence out of the chain complexes $C_*(A \cap B), C_*(A), C_*(B)$, and $C_*(X)$. Taking our cue from the <u>proof</u> of S-vK, we might think of chains in X as sums of chains in A and B, except that we mod out by chains in $A \cap B$. So we might try the sequence

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n(X) \to 0$$

where $j_n : C_n(A) \oplus C_n(B) \to C_n(X)$ is defined as $j_n(a, b) = a + b$. In order to get exactness at the middle term (i.e., image = the kernel of this map, which is $\{(x, -x) : x \in C_n(A) \cap C_n(B)\}$), we set $i_n : C_n(A \cap B) \to C_n(A) \oplus C_n(B)$ to be $i_n(x) = (x, -x)$, since $C_n(A \cap B) = C_n(A) \cap C_n(B)$! i_n is then injective, and we certainly have that this sequence is exact at the middle term. But, in general, j_n is far from surjective! The image of j_n is the set of n-chains that can be expressed as sums of chains in A and B. Which of course not every chain in X can be; singular simplices in X need not map entirely into either A or B.

We can solve this by <u>replacing</u> $C_n(X)$ with the image of j_n, calling it, say, $C_n^{\{A,B\}}(X)$. [Note: these groups <u>would</u> form a chain complex!] Then we have a short exact sequence, and hence a long exact homology sequence. But it involves a "new" homology group $H_n^{\{A,B\}}(X)$. The <u>point</u> is that, like S-vK, under the right conditions, this new homology is the same as $H_n(X)$!

Starting from scratch, the idea is that, starting with an *open cover* $\{\mathcal{U}_\alpha\}$ of X (or, more generally, with a collections of subspaces A_α whose interiors $\mathcal{U}_\alpha$ cover X), we build the *chain groups subordinate to the cover*,

$$C_n^{\mathcal{U}}(X) = \{\sum a_i\sigma_i^n : \sigma_i : n : \Delta^n \to X, \sigma_i^n(\Delta^n) \subseteq \mathcal{U}_\alpha \text{ for some } \alpha\} \subseteq C_n(X).$$

Since the face of any simplex mapping into $\mathcal{U}_\alpha$ also maps into $\mathcal{U}_\alpha$, our ordinary boundary maps induce boundary maps on these groups, turning $(C_n^{\mathcal{U}}(X), \partial_n)$ into a chain complex. Our main result is that the inclusion i of these groups into $C_n(X)$ induces an isomophism on homology. And to show this, we (could) once again use the notion of a chain homotopy.

Theorem: There is a chain map $b : C_n(X) \to C_n^{\mathcal{U}}(X)$ so that $i \circ b$ and $b \circ i$ are both chain homotopic to the identity. i consequently induces isomorphisms on homology.

But we won't prove it quite that way! Another approach is to use the short exact sequence of chain complexes

$$0 \to C_n^{\mathcal{U}}(X) \xrightarrow{i} C_n(X) \to C_n(X)/C_n^{\mathcal{U}}(X) \to 0$$

to build a long exact homology sequence. Every third group is $H_n(C_*(X)/C_*^{\mathcal{U}}(X))$; if we show that these groups are 0, then i_* will be an isomorphism. And to show <u>this</u>, working back through the definition of homology classes in $H_n(C_*(X)/C_*^{\mathcal{U}}(X))$, we need to show that if $z \in C_n(X)$ with $\partial z \in C_{n-1}^{\mathcal{U}}(X)$ (i.e., z is a relative cycle), then there is a $w \in C_{n+1}(X)$ with $z - \partial w \in C_n^{\mathcal{U}}(X)$ (i.e., z is a relative boundary). In words, if z has boundary a sum of small simplices, then there is a chain z' made of small simplices so that $z - z'$ is a boundary.

And the key to building z' and w is a process known as *barycentric subdivision.* The idea is really the same as for S-vK; we cut our singular simplices up into tiny enough pieces so that (via the Lebesgue number theorem) each piece maps into <u>some</u> $\mathcal{U}_\alpha$. Unlike S-vK, though, we want to do this in a more structured way, so that the cutting up process is "compatible" with our boundary maps. And the best way to describe this cutting up is through *barycentric coordinates.* Recall that an n-simplex is the set of convex linar combinations $\sum x_i v_i$ with $x_i \geq 0$ and $\sum x_i = 1$. The map which sends <u>an</u> n-simplex to <u>the</u> n-simplex Δ^n is literally the map $\sum x_i v_i \mapsto (x_0, \dots, x_n)$. These are the barycentric coordinates of an n-simplex. Since, formally, all singular simplices are considered to have Δ^n for their domain, we can describe barycentric subdivision by describing how to cut up Δ^n. The idea is to build a process that is compatible with the boundary map, so that the subdivision, when restricted to a sub-simplex, is the subdivision of that sub-simplex. A 1-simplex $[v_0, v_1]$ is subdivided by adding the barycenter $w = (v + 0 + v_1)/2$ as a vertex, cutting $[v_0, v_1]$ into two 1-simplices ,$[v_0, w]$,$[w, v_1]$. A 2-simplex $[v_0, v_1, v_2]$ will, to be compatible with the boundary map, have its boundary cut into 6 1-simplices; using the barycenter $(v_0 + v_1 + v_2)/3$ we can cone off each of these 1-simplices to subdivide $[v_0, v_1, v_2]$ into 6 2-simplices. Taking the cue that $2 = (1 + 1)!$, $6 = (2 + 1)!$ is probably no accident, we might expect that an n-simplex will be cut into $(n + 1)!$ n-simplices. Note that this is the number of ways of ordering the vertices of our simplex. And following the "pattern" of our two test cases, where each new simplex was the convex span of vertices chosen as (vertex) , (barycenter of a 1-simplex having (vertex) as a vertex), (barycenter of a 2-simplex containing the previous 2 vertices), etc., we are led to the idea that the barycentric subdivision of an n-simplex $[v_0, \dots, v_n]$ is the $(n + 1)!$ n-simplices,

$$[v_{\alpha(0)}, (v_{\alpha(0)} + v_{\alpha(1)})/2, (v_{\alpha(0)} + v_{\alpha(1)} + v_{\alpha(2)})/3, \ldots , (v_{\alpha(0)} + \cdots v_{\alpha(n)})/(n + 1)]$$

one for every permutation α of $\{0, \ldots , n\}$. And since we want to take into account orientations as well, the natural thing to do is to define the barycentric subdivision of a singular n-simplex $\sigma : [v_0, \ldots , v_n] \to X$ to be

$$S(\sigma) = \sum_{\alpha} (-1)^{\mathrm{sgn}(\alpha)} \sigma|_{[v_{\alpha(0)},(v_{\alpha(0)}+v_{\alpha(1)})/2,(v_{\alpha(0)}+v_{\alpha(1)}+v_{\alpha(2)})/3,\ldots,(v_{\alpha(0)}+\cdots v_{\alpha(n)})/(n+1)]}$$

where the sum is taken over all permutations of $\{0, \ldots , n\}$. This (extending linearly over the chain group) is the subdivision operator, $S : C_n(X) \to C_n(X)$. A "routine" calculation establishes that $\partial S = S\partial$, i.e., it is a chain map (i.e., it behaves well on the boundary of our simplices). The point to this operator is that all of the subsimplices in the sum are a definite factor smaller than the original simplex. In fact, if the diameter of $[v_0, \ldots , v_n]$ is d (the largest distance between points, which will, because it is the convex span of the vertices, be the largest distance between vertices), then every individual simplex in $S(\sigma)$ will have diameter at most $nd/(n+1)$ (the result of a little Euclidean geometry and induction). So by repeatedly applying the subdivision operator S to a singular simplex, we will obtain a singular chain $S^k(\sigma)$, which is "really" σ written as a sum of tiny simplices, whose singular simplices have image as small as we want. Or put more succinctly, if $\{\mathcal{U}_\alpha\}$ is an open cover of X and $\sigma : \Delta^n \to X$ is a singular n-simplex, then choosing a Lebesgue number ϵ for the open cover $\sigma^{-1}(\mathcal{U}_\alpha)$ of the compact metric space Δ^n, and choosing a k with $d(n/(n + 1))^k < \epsilon$, we find that $S^k(\sigma)$ is a sum of singular simplices each of which maps into one of the $\mathcal{U}_\alpha$, i.e., $S^k(\sigma) \in C_n^{\mathcal{U}}(X)$.

In the end, we will choose our needed "small" cycle to be $z' = S^k z$. and to show that their difference is a boundary, we will build a chain homotopy between Id and S^k. And to do that, we define a map $R : C_n(X) \to C_{n+1}(X \times I)$; when followed by the projection-induced map $p_\# : C_{n+1}(X \times I) \to C_{n+1}(X)$, we get a map $T : C_n(X) \to C_{n+1}(X)$, and show that $\partial T + T\partial = I - S$. Then we set $H = \sum TS^j$, where the sum is taken over $j = 0, \dots k-1$. Once we define T (!) , we will have $\partial H_k + H_k\partial = \sum \partial TS^j + TS^j\partial = \sum(\partial T + T\partial)S^j = \sum(S^j - S^{j+1}) = I - S^k$ (since the last sum telescopes). And defining R, is, formally, just another particular sum. Setting up some notation, thinking of $\Delta^n \times I$, as before, as having vertices $\{v_0, \dots v_n\}$ on the 0-end and $\{w_0, \dots, w_n\}$ on the 1-end, $N = \{0, \dots, n\}$, $\Pi(Q) =$ the group of permutations of Q, and $\sigma' = \sigma \times I : \Delta^n \times I \to X \times I$), we have

$$R(\sigma) = \sum_{A \subseteq N} \sum_{\pi \in \Pi(N \setminus A)} \{(-1)^{|A|}(-1)^{\mathrm{sgn}(\pi)} \prod_{j \in N \setminus A} (-1)^j\}$$

$$\sigma'|_{[v_{i_0}, \dots, v_{i_j}, (w_{i_0} + \cdots w_{i_j})/(j+1), (w_{i_0} + \cdots w_{i_j} + w_{\pi(i_{j+1})})/(j+2), \dots, (w_{i_0} + \cdots w_{i_j} + w_{\pi(i_{j+1})} + \cdots w_{\pi(i_n)})/(n+1)]}$$

where we sum over all non-empty subsets of $\{0, \dots n\}$ (with the induced ordering on vertices from the ordering on $\{0, \dots, n\}$). Intuitively, this map "interpolates" between the simplex $[v_0, \dots v_n]$ and the barycentric subdivision on $w_0, \dots, w_n$, by taking the (signed sums of the) convex spans of simplices on the bottom (0) and simplices on the top (1). Again, a "routine" calculation will establish that $\partial T + T\partial = I - S$, as desired. [At any rate, I verified it for n=1,2; the formula for the sign of each simplex was determined by working backwards from these examples.]