Exact sequences: Most of the fundamental properties of homology groups are described in terms of exact sequences. A sequence of homomorphisms

$$\ldots \stackrel{f_{n+1}}{\rightarrow} A_n \stackrel{f_n}{\rightarrow} A_{n-1} \stackrel{f_{n-1}}{\rightarrow} a_{n-2} \rightarrow \cdots$$

of abelian groups is called *exact* if $\mathrm{im}(f_n) = \ker(f_{n-1})$ for every n. In most cases, we get the most mileage out of an exact sequence when some of the groups are trivial;

$$0 \rightarrow A \stackrel{f}{\rightarrow} B \text{ is exact} \Leftrightarrow\ f \text{ is injective, and } A \stackrel{f}{\rightarrow} B \rightarrow 0 \text{ is exact} \Leftrightarrow\ f \text{ is surjective.}$$

An exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is called a *short exact sequence.*

The main tool we will use turns a family of short exact sequences of chain maps between three chain complexes into a single *long exact homology sequence.*

Given chain complexes $\mathcal{A} = (A_n, \partial)$, $\mathcal{B} = (B_n, \partial')$, and $\mathcal{C} = (C_n, \partial'')$ and short exact sequences of chain maps (i.e., $\partial' i_n = i_n \partial$, $\partial'' j_n = j_n \partial'$)

$$0 \rightarrow A_n \stackrel{i_n}{\rightarrow} B_n \stackrel{j_n}{\rightarrow} C_n \rightarrow 0$$

there is a general result which provides us with a long exact sequence

$$\cdots \stackrel{\partial}{\rightarrow} H_n(\mathcal{A}) \stackrel{i_*}{\rightarrow} H_n(\mathcal{B}) \stackrel{j_*}{\rightarrow} H_n(\mathcal{C}) \stackrel{\partial}{\rightarrow} H_{n-1}(\mathcal{A}) \stackrel{i_*}{\rightarrow} \cdots$$

Most of the work is in defining the "boundary" map ∂. Given an element $[z] \in H_n(\mathcal{C})$, a representative $z \in C_n$ satisfies $\partial''(z) = 0$. But j_n is onto, so there is a $b \in B_n$ with $j_n(b) = z$, Then $i_{n-1}\partial'(b) = \partial'' j_n(b) = 0$, so $\partial'(b) \in \ker(j_{n-1})$ $=\mathrm{im}(a_{n-1})$. So there is an $a \in A_{n-1}$ with $i_{n-1}(a) = \partial'(b)$. But then $i_{n-2}\partial(a) = \partial' i_{n-1}(a) = \partial'\partial'(b) = 0$, so, since i_{n-2} is injective, $\partial a = 0$, so $a \in Z_{n-1}(\mathcal{A})$, and so represents a homology class $[a] \in H_n(\mathcal{A})$. We define $\partial([z]) = [a]$.

Is this well-defined? We need to show that the class $[a]$ is independent of the choices made along the way. The choice of a isn't really one; i_{n-1} is, by assumption, injective. For b, if $j_n(b) = z = j_n(b')$, then $j_n(b-b') = 0$, so $b - b' = i_n(w)$ for some $w \in A_n$. Then $\partial' b' = \partial' b - \partial' i_n(w) = \partial' b - i_{n-1}\partial(w)$; choosing $a' = a - \partial(w)$ we have $i_{n-1}(a') = \partial'(b')$. But then $[a'] = [a - \partial w] = [a] - [\partial w] = [a]$. Finally, there is a choice of z ; if $[z] = [z']$, then $z' = z + \partial'' w$ for some $w \in C_{n+1}$; but then choosing b', w' with $j_n(b') = z'$, $j_{n+1}(w') = w$, we have $\partial'' w = \partial'' j_{n+1}(w') = j_n \partial'(w')$, so $z' = z + \partial'' w = j_n(b + \partial' w')$, so we may choose $b' = b + \partial' w'$ (since the result is independent of this choice!), then since $\partial' b' = \partial' b$ everything continues the same.

Now to exactness! We need to show three (types of) equalities, which means six containments. Three (image $\subseteq$ kernel) are shown basically by showing that compositions of two consecutive homomorphisms are trivial. $j_n i_n = 0$ immediately implies $j_* i_* = 0$. From the definition of ∂, $i_* \partial[z] = [i_n(a)] = [\partial'(b)] = 0$, and $\partial j_*[z] = \partial[j_n(z)] = [a]$, where $i_{n-1}(a) = \partial'(z) = 0$, so $a = 0$ (since i_{n-1} is injective), so $[a] = 0$.

For the opposite containments, if $j_*[z] = [j_n(z)] = 0$, then $j_n(z) = \partial'' w$ for some w. Since j_{n+1} is onto, $w = j_{n+1}(b)$ for some b. Then $j_n(z - \partial' b) = \partial'' w - \partial'' j_{n+1} b = 0$, so $z = \partial' b = i_n(a)$ for some a, so $i_*[a] = [z - \partial' b] = [z]$. So $\ker j_* \subseteq \mathrm{im} i_*$. If $i_*[z] = 0$, then $i_n(z) = \partial' w$ for some $w \in B_{n+1}$. Setting $c = j_{n+1}(w)$, then $\partial'' c = j_n \partial' w - i_n i_n(z) = 0$, so $[c] \in H_{n+1}(\mathcal{C})$, and computing $\partial[c]$ we can choose w for the 1st step and z for the 2nd, so $\partial[c] = [z]$. So $\ker j_n \subseteq \mathrm{im}\partial$. Finally, if $\partial[z] = 0$, then $z = j_n(b)$ for some b, and $\partial' b = i_{n-1}(a)$ with $[a] = 0$, i.e., $a = \partial w$ for some w. So $\partial' b = i_{n-1}\partial w = \partial' i_n w$. But then $\partial'(b - i_n w) = 0$, and $j_n(b - i_n w) = z - 0 = z$, so $z \in \mathrm{im}(j_n)$, so $[z] \in \mathrm{im}(j_*)$. So $\ker \partial \subseteq \mathrm{im}(j_n)$. Which finishes the proof!

Now all we need are some new chain complexes!