Some applications:

Given a free group $G = F(a_1, \ldots a_n)$ and a collection of words $w_1, \ldots w_m \in G$, we can determine the rank and index of the subgroup it H they generate by building the corresponding cover. The idea is to start with a bouquet of m circles, each subdivided and labelled to spell out the words w_i. Then we repeatedly identify edges sharing on common vertex if they are labeled precisely the same (same letter *and* same orientation). This process is known as *folding*.

One can inductively show that the (obvious) maps from these graphs to the bouquet of n circles X_n both have image H under the induced maps on π_1; since the map for the unfolded graph factors through the one for the folded graph, the image from the folded graph can only get smaller, but we can still spell out the same words as loops in the folded graph, so the image can also only have gotten bigger! We continue this folding process until there is no more folding to be done; the resulting graph X is what is known (in combinatorics) as a *graph covering*; the map to X_n is locally injective. If this map is a covering map, then our subgroup H has finite index (equal to the degree of the covering) and we can compute the rank of H (and a basis!) from the folded graph. If it is not a covering map, then the map is not locally surjective at some vertices; if we graft trees onto these vertices, we can extend the map to an (infinite-sheeted) covering map without changing the homotopy type of the graph. H therefore has infinite index in G, and its rank can be computed from $H \cong \pi_1(X)$.

Note that for a graph Γ to be a covering of another graph, with k sheets, say, the number of vertices and edges of Γ must both be a mulitple of k. This little observation can be very useful when trying to decide what graphs Γ might cover!

Kurosh Subgroup Theorem: If $H < G_1 * G_2$ is a subgroup of a free product, then H is (isomorphic to) a free product of a collection of conjugates of subgroups of G_1 and G_2 and a free froup. The proof is to build a space by taking 2-complexes X_1 and X_2 with π_1's isomorphic to G_1, G_2 and join their basepoints by an arc. The covering space of this space X corresponding to H consists of spaces that cover X_1, X_2 (giving, after basepoint considerations, the conjugates) connected by a collection of arcs (which, suitably interpreted, gives the free group).

Residually finite groups: G is said to be residually finite if for every $g \neq 1$ there is a finite group F and a homomorphism $\varphi : G \to F$ with $\varphi(g) \neq 1$ in F. This amounts to saying that $g \notin$ the (normal) subgroup $\ker(\varphi)$, which amounts to saying that a loop corresponding to g does <u>not</u> lift to a loop in the finite-sheeted covering space corresponding to $\ker(\varphi)$. So residual finiteness of a group can be verified by building coverings of a space X with $\pi_1(X) = G$. For example, free groups can be shown to be residually finite in this way.

Ranks of free (sub)groups: A free group on n generators is isomorphic to a free group on m generators $\Leftrightarrow$ $n = m$; this is because the abelianizations of the two groups are $\mathbb{Z}^n, \mathbb{Z}^m$. The (minimal) number of generators for a free group is called its *rank*.

Postscript: why care about covering spaces? The preceding discussion probably makes it clear that covering places play a central role in (combinatorial) group theory. It also plays a role in embedding problems; a common scenario is to have a map $f : Y \to X$ which is injective on π_1, and we wish to know if we can lift f to a finite-sheeted covering so that the lifted map $\widetilde{f}$ is homotopic to an embedding. Information that is easier to obtain in the case of an embedding can then be passed down to gain information abut the original map f. And covering spaces underlie the theory of analytic continuation in complex analysis; starting with a domain $D \subseteq \mathbb{C}$, what analytic continuation really builds is an (analytic) function from a covering space of D to $\mathbb{C}$. For example, the logarithm is really defined as a map from the universal cover of $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$. The various "branches" of the logarithm refer to which sheet in this cover you are in.