Some examples:

Fundamental groups of graphs: Every finite connected graph Γ has a *maximal tree* T, a connected subgraph with no simple circuits. Since any tree is the union of smaller trees joined at a vertex, we can, by induction, show that $\pi_1(T) = \{1\}$. In fact, if e is an outermost edge of T, then T deformation retracts to $T \setminus e$, so, by induction, T is contractible. Consequently (*Hatcher, Proposition 0.17*), Γ and the quotient space Γ/T are homotopy equivalent, and so have the same π_1. But $\Gamma/T = \Gamma_n$ is a bouquet of n circles for some n. If we let $\mathcal{U} =$ a neighborhood of the vertex in Γ_n, which is contractible, then, by singling out one petal of the bouquet, we have

$$\Gamma_n = (\Gamma_{n-1} \cup \mathcal{U}) \cup (\Gamma_1 \cup \mathcal{U}) = X_1 \cup X_2$$

with $\Gamma_k \cup \mathcal{U} \sim (\Gamma_k \cup \mathcal{U})/\mathcal{U} \cong \Gamma_k$. Since $X_1 \cap X_2 = \mathcal{U} \sim *$, we have

$$\pi_1(\Gamma_n) \cong \pi_1(\Gamma_{n-1}) *_1 \pi_1(\Gamma_1) = \pi_1(\Gamma_{n-1}) * \mathbb{Z}$$

So, by induction, $\pi_1(\Gamma) \cong \pi_1(\Gamma_n) \cong \mathbb{Z} * \cdots * \mathbb{Z} = F(n)$, the free group on n letters, where $n =$ the number of edges not in a maximal tree for Γ. The generators for the group consist of the edges not in the tree, prepended and appended by paths to the basepoint.

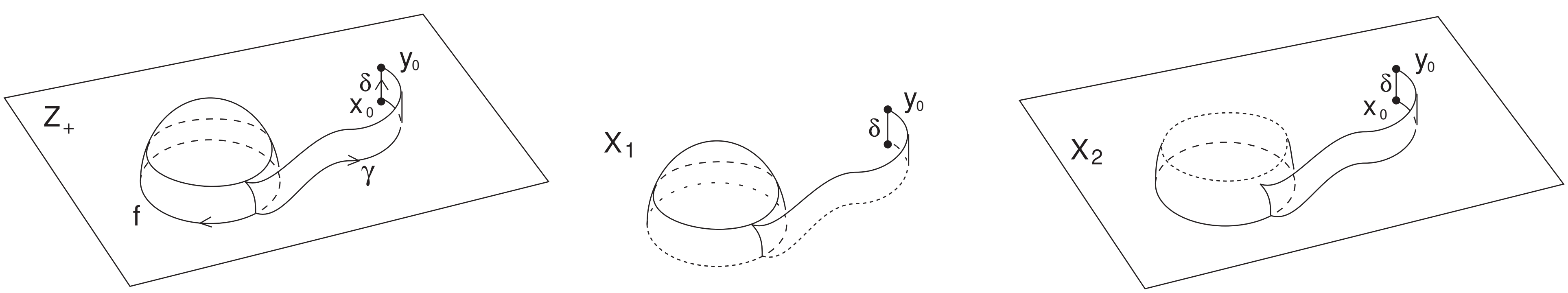
Gluing on a 2-disk: $f : \partial\mathbb{D}^2 \to X$ continuous, then we construct the quotient space $Z = (X \coprod \mathbb{D}^2)/\{x \simeq f(x) : x \in \partial\mathbb{D}^2\}$, the result of gluing $\mathbb{D}^2$ to X along f.

We can use Seifert - van Kampen to compute π_1 of the resulting space; if we wish to be careful with basepoints x_0, we include a rectangle R, the mapping cylinder of a path γ running from $f(1,0)$ to x_0, glued to $\mathbb{D}^2$ along the arc from $(1/2,0)$ to $(1,0)$ (see figure). This space Z_+ deformation retracts to Z; it is simpler work with the basepoint y_0 lying above x_0.

Write $D_1 = \{x \in \mathbb{D}^2 : ||x|| < 1\} \cup (R \setminus X)$ and $D_2 = \{x \in \mathbb{D}^2 : ||x|| > 1/3\} \cup R$, then we can write $Z_+ = D_+ \cup (X \cup D_2) = X_1 \cup X_2$. But since $X_1 \simeq *$, $X_2 \simeq X$ (it is essentially the mapping cylinder of the maps f and γ) and $X_1 \cap X_2 = \{x \in \mathbb{D}^2 : 1/3 < ||x|| < 1\} \cap (R \setminus X) \simeq S^1$, we find that

$$\pi_1(Z, y_0) \cong \pi_1(X_2, y_0) *_{\mathbb{Z}} \{1\} = \pi_1(X_2)/ < \mathbb{Z} >^N \cong \pi_1(X_2)/ < [\overline{\delta} * \overline{\gamma} * f * \gamma * \delta] >^N$$

Use δ for a change of basepoint isomorphism, and then a homotopy equivalence from X_2 to X (fixing x_0), we have, if $\pi_1(X, x_0) =< \Sigma|R >$, then $\pi_1(Z) =< \Sigma|R \cup \{[\overline{\gamma} * f * \gamma]\} >$. So the effect of gluing on a 2-disk on the π_1 is to add a new relator, namely the word represented by the attaching map (adjusting for basepoint).



Wirtinger presentations for knot complements:

A *knot* K is (the image of) an embedding $h : S^1 \hookrightarrow \mathbb{R}^3$. Wirtinger gave a prescription for taking a planar projection of K and producing a presentation of $\pi_1(\mathbb{R}^3 \setminus K)$. The idea: think of K as lying on the projection plane, except near the crossings, where it arches over and dives under. Express $\mathbb{R}^3 \setminus N(K) \simeq X$ as the union of $C_- = X \cap \{(x, y, z \in \mathbb{R}^3 : z \leq 0\}$ and $C_+ = X \cap \{(x, y, z \in \mathbb{R}^3 : z \geq 0\}$, with intersection $(\mathbb{R}^2 \times \{0\}) \setminus N(K)$.
But each of C_-, C_+ is $\simeq$ a bouquet of n circles, where $n =$ the number of crossings in the projection of K.