Seifert-van Kampen Theorem:

If $X = \mathcal{U} \cup \mathcal{V}$ with $\mathcal{U}, \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \mathcal{W}$ open and path-connected, and $x_0 \in \mathcal{W}$, then $\pi_1(X, x_0)$ is the pushout of $\pi_1(\mathcal{U})$ and $\pi_1(\mathcal{V})$ along $\pi_1(\mathcal{W})$.

The basic idea: a proof in two (and a half) parts.

The inclusions $i_{\mathcal{U}} : \mathcal{U} \to X$ and $i_{\mathcal{V}} : \mathcal{V} \to X$ induce a homomorphism

$\varphi : \pi_1(\mathcal{U}) * \pi_1(\mathcal{V}) \to \pi_1(X)$.

First step: show that φ is surjective (using a Lebesgue number argument!). We also have inclusion-induced homomorphisms from the maps

$j_{\mathcal{U}} : \mathcal{W} \to \mathcal{U}$ and $j_{\mathcal{V}} : \mathcal{W} \to \mathcal{U}$.

Second step: show that

$\ker(\varphi) = \langle \{ j_{\mathcal{U}*}(\gamma) j_{\mathcal{V}*}(\gamma^{-1}) : \gamma \in \pi_1(W) \} \rangle^N$

(using a Lebesgue number argument!). Therfore, φ induces an isomorphism

$\vartheta : \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{W})} \pi_1(\mathcal{V}) \to \pi_1(X)$, as desired.

The reliance upon a decomposition of X into open sets in the theorem is dictated by our use of Lebesgue numbers. In practice, we sidestep this (typically annoying) condition, by decomposing into closed sets $X = C \cup D$, but insist that these sets have open neighborhoods $\mathcal{U}, \mathcal{V}$ so that $\mathcal{U}, \mathcal{V}, \mathcal{U} \cap \mathcal{V}$ deformation retract to $C, D, C \cap D$ respectively. The closed sets therefore have (essentially) the same fundamental groups as the open sets, and the analogous result follows. Later we will show how we can always arrange this hypothesis for most "reasonable" closed subsets of "reasonable" spaces.

Proof, part one:

To show that $\varphi : \pi_1(\mathcal{U}) * \pi_1(\mathcal{V}) \to \pi_1(X)$ is surjective, we wish, given a loop $\gamma : (I, \partial i) \to (X, x_0)$, to show that γ is homotopic rel endpoints to the concatenation of loops which each map into either $\mathcal{U}$ or $\mathcal{V}$. But $\mathcal{U}, \mathcal{V}$ form an open cover of X, so there is a Lebesgue number $1/n$ for the cover $\gamma^{-1}(\mathcal{U}), \gamma^{-1}(\mathcal{V})$ of I. Partitioning I into n equal pieces, and writing $\gamma_i = \gamma|_{[\frac{i-1}{n}, \frac{i}{n}]}$, each γ_i maps into $\mathcal{U}$ or $\mathcal{V}$. As in our proof of $\pi_1(S^1) \cong \mathbb{Z}$, we amalgamate the subintervals until the subsets they map into alternate between the two as we traverse the interval I.

Calling the resulting intervals $I_j = [z_{j-1}, z_j]$, $j = 1, \dots, m$, we then have that $\gamma(z_j) \in \mathcal{U} \cap \mathcal{V}$ for every j. Since $\mathcal{U} \cap \mathcal{V}$ is path-connected, we can find a path α_j in $\mathcal{U} \cap \mathcal{V}$ from z_j to x_0. We will recycle the notation $\gamma_j = \gamma|_{[z_{j-1}, z_j]}$; then

$$\begin{aligned}
&\gamma \simeq \gamma_0 * \cdots * \gamma_m \simeq \gamma_0 * (\alpha_1 * \overline{\alpha_1}) * \gamma_1 * \cdots * \gamma_{m-1} * (\alpha_{m-1} * \overline{\alpha_{m-1}}) * \gamma_m \\
&\simeq (\gamma_0 * \alpha_1) * (\overline{\alpha_1} * \gamma_1 * \alpha_2) * \cdots * (\overline{\alpha_{m-2}} * \gamma_{m-1} * \alpha_{m-1}) * (\overline{\alpha_{m-1}} * \gamma_m) \\
&= \eta_0 * \cdots * \gamma_m,
\end{aligned}$$

which is a concatenation of loops (based at x_0) which alternately map into $\mathcal{U}$ and $\mathcal{V}$.

To be completely pedantic, if we let $\omega_j = \eta_j$ with its codomain changed from X to $\mathcal{U}$ or $\mathcal{V}$ as appropriate (the ω_i are continuous, since restriction of codomain preserves continuity (using subspace topologies)), then $[\omega_0] \cdot \cdots \cdot [\omega_m] \in \pi_1(\mathcal{U}) * \pi_1(\mathcal{V})$, and $\varphi([\omega_0] \cdot \cdots \cdot [\omega_m]) = [\eta_0] \cdot \cdots \cdot [\eta_m] = [\eta_0 * \cdots * \eta_m] = [\gamma]$, so φ is surjective, as desired.

Proof, part two:

It remains to show that $\ker(\varphi) = \langle\{j_{\mathcal{U}*}(\gamma)j_{\mathcal{V}*}(\gamma^{-1}) : \gamma \in \pi_1(W)\}\rangle^N = \mathcal{H}$.

The containment $\supseteq$ follows by showing that $\varphi(j_{\mathcal{U}*}(\gamma)j_{\mathcal{V}*}(\gamma^{-1})) = 1$ in $\pi_1(X)$; but $\varphi(j_{\mathcal{U}*}(\gamma)j_{\mathcal{V}*}(\gamma^{-1})) = i_{\mathcal{U}*}j_{\mathcal{U}*}(\gamma) \cdot i_{\mathcal{V}*}j_{\mathcal{V}*}(\gamma^{-1})) = (i_{\mathcal{U}} \circ j_{\mathcal{U}})_*(\gamma) \cdot (i_{\mathcal{V}} \circ j_{\mathcal{V}})_*(\gamma^{-1})$. But since both $(i_{\mathcal{U}} \circ j_{\mathcal{U}})$ and $(i_{\mathcal{V}} \circ j_{\mathcal{V}})$ are equal to the inclusion map $\iota : \mathcal{U} \cap \mathcal{V} \to X$, we have $\varphi(j_{\mathcal{U}*}(\gamma)j_{\mathcal{V}*}(\gamma^{-1})) = \iota_*(\gamma) \cdot \iota_*(\gamma^{-1}) = \iota_*(\gamma \cdot \gamma^{-1}) = \iota_*(1) = 1$.

For the opposite containment, suppose that $\varphi([\gamma_1] \cdot \dots \cdot [\gamma_n]) = [\gamma_1 * \dots * \gamma_n] = 1$ where each γ_i maps into $\mathcal{U}$ or $\mathcal{V}$. Then we have a homotopy rel endpoints $H : I \times I \to X$ from $\gamma_1 * \dots * \gamma_n$ to the constant map at x_0. We wish to show that $[\gamma_1] \cdot \dots \cdot [\gamma_n]$ is equal, in $\pi_1(\mathcal{U}) * \pi_1(\mathcal{V})$, to a product of conjugates of elements of the form $j_{\mathcal{U}*}(\gamma)j_{\mathcal{V}*}(\gamma^{-1})$.

As before, we find a Lebesgue number $\epsilon > 0$ for the open cover $H^{-1}(\mathcal{U}), H^{-1}(\mathcal{V})$ of $I \times I$, so that there is an N so that every $\frac{1}{N} \times \frac{1}{N}$ subsquare in $I \times I$ maps into $\mathcal{U}$ or $\mathcal{V}$ under H. Partitioning $I \times I$ into N^2 squares, these squares form N horizontal strips, each of height $1/N$.

The proof proceeds by showing that each of the loops $\alpha_i : t \mapsto H(t, (N-i)/N)$ lies in $\mathcal{H}$, by induction on i; the initial case $i = 0$ is the constant loop, which is immediate, while the case $i = N$ is our desired result. For the inductive step we show that if the bottom of one of our horizontal strips lies in $\mathcal{H}$ then the top does, as well.

The one delicate point in what follows is that $\gamma_1 * \dots * \gamma_n$ is given as an explicit product of loops in $\mathcal{U}, \mathcal{V}$, and we must remember in our deformations to always be dealing with loops into these two sets, in order to show that <u>in the free product</u> $\gamma_1 * \dots * \gamma_n$ is a product of conjugates of the form we desire.

We have our domain cut into an $n \times n$ grid, so that each subsquare maps into $\mathcal{U}$ or $\mathcal{V}$; pick one for each. As before, we amalgamate horizontally adjacent squares if they are labeled the same; then each horizontal strip is cut into rectangles which alternate their label. The vertical edges of the rectangles then map into $\mathcal{U}) \cap \mathcal{V}$, and so their endpoints do, as well. Join each of these endpoints to our basepoint by paths $\eta_{i,j}$. Then the top and bottoms of the strips are
$\gamma_{i,1} * \cdots \gamma_{i,m} \simeq \gamma_{i,1} * (\eta_{i,1} * \overline{\eta_{i,1}}) * \gamma_{i,2} * \cdots * \gamma_{i,m-1} * (\eta_{i,m-1} * \overline{\eta_{i,m-1}}) * \gamma_{i,m}$
$\simeq (\gamma_{i,1} * \eta_{i,1}) * (\overline{\eta_{i,1}} * \gamma_{i,2} * \eta_{i,2}) * \cdots * \gamma_{i,m-1} * \eta_{i,m-1}) * (\overline{\eta_{i,m-1}} * \gamma_{i,m})$ is a product of loops in $\mathcal{U})$ and $\mathcal{V}$.

Our <u>real</u> inductive hypothesis is that the bottom of thr strip has a partition and collection of paths to the basepoint in $\mathcal{U} \cap \mathcal{V}$ from those partition points so that the resulting loops (as above), as an element of $\pi_1(\mathcal{U})) * \pi_1(\mathcal{V})$, lies in $\mathcal{H}$. (Note that this is immediate for the bottom of the square; it is literally written as products of loops and their inverses (since we go up and down each path).) The first point is that the partitioning coming from the strip and the paths chosen there <u>also</u> express the bottom edge as an element of $\mathcal{H}$. This is because we can add the paths from one to the other without changing the element (the added paths map into $\mathcal{U} \cap \mathcal{V}$, so change nothing, up to homotopy), but then when we change perspectives between the two sets of paths some of the sub-edges may change their label, and these subedges then map into $\mathcal{U} \cap \mathcal{V}$. But this change of label is precisely the same as inserting an element of the form $\alpha = j_{\mathcal{U}*}(\gamma)^{-1} j_{\mathcal{V}*}(\gamma)$ (or its inverse) into our group element; multiplication by α literally removes $j_{\mathcal{U}*}(\gamma)$ and replaces it with $j_{\mathcal{V}*}(\gamma)$; its inverse does the reverse.

Dealing with the horizontal strip itself is more straightforward. The top of the strip is homotopic to the bottom, and is equal, in $\pi_1(\mathcal{U})) * \pi_1(\mathcal{V})$ to the bottom strip with pairs of loops inserted, namely (top path)*(vertical edge)*(bottom path) and its inverse. These inserted loops all map into $\mathcal{U} \cap \mathcal{V}$, but it is <u>thought</u> <u>of</u> as one copy mapping into $\mathcal{U}$ and the other into $\mathcal{V}$, to match the subset that each subrectangle maps into. Again, this amounts to inserting an element of $\mathcal{H}$ into an element of $\pi_1(\mathcal{U})) * \pi_1(\mathcal{V})$ which, by the inductive hypothesis, also lies in $\mathcal{H}$. Therefore the top, partitioned as the subrectangles dictate, lies in $\mathcal{H}$.

This insertion process leaves us in $\mathcal{H}$, because:

If $uv \in \mathcal{H}$ and $w \in \mathcal{H}$, then $uwv = (uwu^{-1})(uv) \in \mathcal{H}$, since $\mathcal{H}$ is normal, so $uwu^{-1} \in \mathcal{H}$.

By induction, we are done: $\ker(\varphi) \subseteq \mathcal{H}$, so $\ker(\varphi) = \mathcal{H}$, as desired.