$\pi_1(S^1) \cong \mathbb{Z}$ **: A proof in two parts**

Show $\mathbb{Z}$ surjects onto $\pi_1(S^1)$, and show that the surjection is injective.

$\mathcal{U}_- = \{(x,y) \in S^1 : y < \epsilon\}, \mathcal{U}_+ = \{(x,y) \in S^1 : y > -\epsilon\}$ open cover of $S^1 \subseteq \mathbb{R}^2 = \mathbb{C}$
$\gamma : (I, \partial I) \to (S^1, 1)$, $1/n$ a Lebesgue number for $\gamma^{-1}(\mathcal{U}_-), \gamma^{-1}(\mathcal{U}_+)$; $I_j = [\frac{j}{n}, \frac{j+1}{n}]$
So $\gamma(I_j) \subseteq \mathcal{U}_\pm$; pick one and label each subinterval $+$ or $-$. If consecutive intervals have the same sign $\Rightarrow$ (**) take union, label with the same sign. Induction $\Rightarrow$ eventually consecutive intervals have opposite signs.

$\mathcal{U}_\pm \cong \mathbb{R}$ and $\pi_1(\mathbb{R}) = 1 \Rightarrow \gamma_j = \gamma|_{I_j} \simeq$ arc in $\mathcal{U}_\pm$ between endpoints. Replace each γ_j with the arc, get loop $\simeq \gamma$ that is a (finite) concatenation of arcs α_j.

If $\alpha_j(0), \alpha_j(1)$ lie in same component of $\mathcal{U}_- \cap \mathcal{U}_+$, then α_j maps into $\mathcal{U}_- \cap \mathcal{U}_+$; change label and apply (**) to lower number of intervals.

Eventually, all α_j "cross" their $\mathcal{U}_\pm$; after a small homotopy, we may assume $\alpha_j(\partial I_j) \subseteq \{-1, 1\}$. Reparametrize so that $I_j = [\frac{j}{m}, \frac{j+1}{m}]$, then $\alpha_j(t) = (\pm\cos(m\pi t), \pm\sin(m\pi t))$ for some choices of $\pm$'s. No backtracking $\Rightarrow m$ is even and $\alpha_j(t) = (\cos(m\pi t), \sin(m\pi t))$ for all j or $\alpha_j(t) = (\cos(m\pi t), -\sin(m\pi t))$ for all j.

Therefore $\alpha \simeq \gamma$ is $\alpha(t) = (\cos(m\pi t), \sin(m\pi t))$ for some (positive, negativce, or zero) m. The map $\varphi : \mathbb{Z} \to \pi_1(S^1)$ given by $m \mapsto [\beta_m : t \mapsto (\cos(m\pi t), \sin(m\pi t))]$ is surjective, by the above, and a homomorphism, since $\beta_m * \beta_n \simeq \beta_{m+n}$.

Second part: φ is injective. I.e., $\beta_m \simeq \beta_n \Rightarrow m = n$.

Introduce *winding number* $w : \pi_1(S^1) \to \mathbb{Z}$. Let $\gamma : (I, \partial I) \to (S^1, 1)$.
$\mathcal{U}_{x+} = \{(x,y) \in S^1 : x > 0\}, \mathcal{U}_{x-} = \{(x,y) \in S^1 : x < 0\}$,
$\mathcal{U}_{y+} = \{(x,y) \in S^1 : y > 0\}, \mathcal{U}_{y-} = \{(x,y) \in S^1 : y < 0\}$: cover of S^1.
$1/n$ = Lebesgue number for their inverse images under γ, $I_j = [\frac{j}{n}, \frac{j+1}{n}]$ (again). $\gamma(\frac{j}{n}), \gamma(\frac{j+1}{n})$ both lie in one of the sets $\Rightarrow$ there is a well-defined (signed) angle θ_j (strictly between $-\pi$ and π) between them. Define $w(\gamma) = \sum \theta_j$. Then show:

(1): $w(\gamma)$ is independent of the partition of I used to compute it. Typical trick: show each is the same as the number computed using the union of the two partitions (by noting that it is unchanged when a single point is added to the partition).

(2): If $\gamma \simeq \beta$, then $w(\gamma) = w(\beta)$. Another standard trick: the homotopy is a concatenation of "small" homotopies. Choose a Lebesgue number $1/m$ for the inverse images of the sets under the homotopy H, and partition $I \times I$ into an m-by-m grid, with vertices (x_i, x_j). Each small square maps into one of our sets, so if $\theta_{i,j}$ =angle from $H(x_i, x_j)$ to $H(x_{i+1}, x_j)$, and $\phi_{i,j}$ =angle from $H(x_i, x_j)$ to $H(x_i, x_{j+1})$, then (walking around a grid square) $\theta_{i,j} + \phi_{i+1,j} = \phi_{i,j} + \theta_{i,j+1}$. This in turn implies that $w(H|_{I \times x_j}) = \sum \theta_{i,j} = \sum \theta_{i,j+1} = w(H|_{I \times x_{j+1}})$. (telescoping sum!). So $w(\gamma) = w(\beta)$.
(3): $w(\beta_n) = n$ (direct computation).

So if $\beta_m \simeq \beta_n$, then $m = n$. So $\varphi(m) = \varphi(n) \Rightarrow m = n$, as desired.