Math 872 Algebraic Topology

Problem Set # 8

Starred (*) problems due Thursday, April 13

39. Giving the n-simplex $X = \Delta^n$ its standard Δ-complex structure, show that the k-skeleton of X has homology

$$\widetilde{H}_i(X^{(k)}) = \begin{cases} \mathbb{Z}^{r(k,n)} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

where $r(k,n) = \binom{n}{k+1}$.

(Hint: our computations preliminary to the proof of "cellular = singular" will help.)

(*) 40. Show that if the sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$$

is exact, then $\sum(-1)^i \mathrm{r}ank\ C_i = 0$. (Hint: pretend it's a chain complex...)

(*) 41. Show that if $\{\mathcal{U}, \mathcal{V}\}$ is an open cover of X and $\mathcal{U} \cap \mathcal{V}, \mathcal{U}, \mathcal{V}$ and X all have $\oplus_i H_i(\text{blah})$ of finite rank, then $\chi(X) = \chi(\mathcal{U}) + \chi(\mathcal{V}) - \chi(\mathcal{U} \cap \mathcal{V})$.

42. Show that if $\widetilde{X}$ is an n-sheeted covering space of the finite CW-complex X, then $\chi(\widetilde{X}) = n\chi(X)$. Conclude that the only non-trivial finite group that can act on an even-dimensional sphere S^{2k} without fixed points is $\mathbb{Z}_2$. (Skip the hard part: the quotient by the group action is a CW-complex...)

43. Show that $H_i(X \times S^n) \cong H_i(X) \oplus H_{i-n}(X)$ for every i; here $H_i(X) = 0$ if $i < 0$.

[One approach: show that $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times D^n_+)$ (Problem #30 will help), and that $H_i(X \times S^n, X \times D^n_+) \cong H_{i-1}(X \times S^{n-1}, X \times D^{n-1}_+)$ by excision and the long exact sequence of the triple $(X \times D^{n-1}_+, X \times S^{n-1}, X \times D^n_-)$.

Hatcher, p.158, # 36 gives a different approach.]