Math 871 Problem Set 2 (revised)

Starred (*) problems are due Thursday, Sept. 8.

(*) 8. If $d : X \times X \to \mathbb{R}$ is a metric on X (i.e., (X, d) is a metric space), show that $\overline{d} : X \times X \to \mathbb{R}$ defined by

$$\overline{d}(x, y) = \min\{d(x, y), 1\} = \begin{cases} d(x, y) & \text{if } d(x, y) \leq 1 \\ 1 & \text{if } d(x, y) \geq 1 \end{cases}$$

is also a metric on X. Show that the two metric spaces (X, d) and $(X, \overline{d})$ have the same open sets.

9. For a set X, and (fixed) $a \in X$, the excluded point topology on X is the collection of subsets $\mathcal{T}_a = \{\mathcal{U} \subseteq X \ : \ a \notin \mathcal{U}\} \cup \{X\}$. Show that $\mathcal{T}_a$ is a topology on X.

(*) 10. Show that the set $\mathcal{T} = \{(a, \infty) \ : \ a \in \mathbb{R}\} \cup \{[a, \infty) \ : \ a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ of subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$.

[Note that $\mathcal{T}$ can be thought of as the sets $\mathcal{U} \subseteq \mathbb{R}$ so that whenever $a \in \mathcal{U}$ and $b \geq a$, then $b \in \mathcal{U}$.]

11. (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is an increasing function [$a \geq b \ \Rightarrow f(a) \geq f(b)$], then, using the topology $\mathcal{T}$ from Problem #10, as a function from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}, \mathcal{T})$, f is continuous.

(b) Show, conversely, that if $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$ is continuous, then f is increasing!
[Consequently, this topology captures increasing functions precisely as its continuous functions...]
[Hint: suppose f isn't increasing: show it is not continuous.]

12. [Munkres, p.83, #13.3] (a) Show that for any set X, the sets
$\{\mathcal{U} \subseteq X \ : \ X \setminus \mathcal{U}$ is countable$\} \cup \{\emptyset\}$ form a topology on X.

(b) Do the sets $\{\mathcal{U} \subseteq X \ : \ X \setminus \mathcal{U}$ is infinite$\} \cup \{\emptyset\}$ always form a topology on X ? Explain why or why not.

(*) 13. [Munkres, p.83, #13.4(a)] If $\mathcal{T}_\alpha$ are all topologies on the same set X, show that $\bigcap_\alpha \mathcal{T}_\alpha$ (the intersection of all of the topologies) is also a topology on X. Is $\bigcup_\alpha \mathcal{T}_\alpha$ (their union) a topology on X?

14. With the (excluded point) topology $\mathcal{T}_a$ on $\mathbb{R}$ from problem #9, and $\mathcal{T}'$ the "usual" topology on $\mathbb{R}$ (open sets are unions of neighborhoods), show that every continuous function $f : (\mathbb{R}, \mathcal{T}_a) \to (\mathbb{R}, \mathcal{T}')$ must be constant.

[Hint: Suppose not! Show that f can't be continuous.]

15. With the topologies from problem #14, show that there do exist continuous functions $f : (\mathbb{R}, \mathcal{T}') \to (\mathbb{R}, \mathcal{T}_a)$ which are not constant.