Math 871 Problem Set 8

Starred (*) problems are due Thursday, November 6.

49. Show that if $X_\alpha, \alpha \in I$ are all path-connected, then so is $\prod_{\alpha\in I} X_\alpha$, if we use the product topology.

(*) 50. Show that if $A_\alpha \subseteq X, \alpha \in I$ are all path-connected, and for some $\alpha_0 \in I$ we have $A_\alpha \cap A_{\alpha_0} \neq \emptyset$ for all $\alpha \in I$, then $\bigcup\limits_{\alpha\in I} A_\alpha$ is path-connected.

51. Show that if $\mathcal{T}$ and $\mathcal{T}'$ are toplogies on X, $(X,\mathcal{T}')$ is compact, and $\mathcal{T} \subseteq \mathcal{T}'$, then $(X,\mathcal{T})$ is compact.

52. Give an example of a space $(X,\mathcal{T})$ and subsets $A, B \subseteq X$ so that A and B are compact but $A \cap B$ is not.

(Note: your space X cannot be Hausdorff....)

(*) 53. Let $X = \mathbb{R}$ with the infinite ray topology

$$\mathcal{T} = \{(a,\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

Show that $A = \{0\}$ is a compact subset of X, but its closure $\overline{A}$ isn't.

54. [Munkres, p.171, Problem #5] Show that if $(X,\mathcal{T})$ is a Hausdorff space and $A, B \subseteq X$ are disjoint compact subsets of X, then there are subsets $U, V \in \mathcal{T}$ so that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

55. [Munkres, p.171, Problem #6] Show that if $f : (X,\mathcal{T}) \to (Y,\mathcal{T}')$ is continuous and surjective, $(X,\mathcal{T})$ is compact, and $(Y,\mathcal{T}')$ is Hausdorff, then f is a closed map (and therefore a quotient map).

(*) 56. Let $(X,\mathcal{T})$ be a Hausdorff space and let

$$\mathcal{T}' = \{U \subseteq X \ : \ X \setminus U \subseteq X \text{ is compact}\} \cup \{\emptyset\}.$$

Show that $\mathcal{T}'$ is a topology on X, and is coarser than $\mathcal{T}$. Show that, in general, the two topologies need not be equal. Do $(X,\mathcal{T})$ and $(X,\mathcal{T}')$ have the same compact subsets? [Note that problem #51 says something about this...]

[This problem (except for the last part) was lifted from an old qualifying exam...]