Math 871 Problem Set 3

Starred (*) problems are due Thursday, Sept. 18.

13. [A useful openness test.] If $(X, \mathcal{T})$ is a topological space, and $A \subseteq X$, show that $A \in \mathcal{T}$ if and only if

$$\text{for all } x \in A, \text{ there is a } U \in \mathcal{T} \text{ so that } x \in U \subseteq A$$

14. Show that if $f, g : X \to Y$ are two functions from the topological space $(X, \mathcal{T})$ to Y, then the finest topology on Y which makes both functions continuous is the intersection of the finest topologies making each function alone continuous.

(*) 15. Show that if $\mathcal{B}$ and $\mathcal{B}'$ are both bases for topologies on X, then so is $\mathcal{B} \cap \mathcal{B}'$, but $\mathcal{B} \cup \mathcal{B}'$ <u>need</u> <u>not</u> <u>be</u>.

16. [Munkres, p.83, Problem #5] Show that if $\mathcal{B}$ is a basis for a topology on X, then $\mathcal{T}(\mathcal{B})$ is the intersection of all topologies that contain $\mathcal{B}$. Show that the analogous statement is true for the topology generated by a subbasis.

(*) 17. Show that, for any set X, the set

$$\mathcal{B} = \{B \subseteq X : X \setminus B \text{ is infinite }\} \cup \{X\}$$

is a basis for a topology on X. What (familiar!) topolog(ies) does it generate?

18. [Munkres, p.83, Problem #8] (a) Show that the collection $\mathcal{B} = \{(a, b) \ : \ a, b \in \mathbb{Q}\}$ is a basis, which generates the 'usual' topology on $\mathbb{R}$.

(b) Show, by contrast, that the collection $\mathcal{B}' = \{[a, b) \ : \ a, b \in \mathbb{Q}\}$ is a basis, but the topology it generates is strictly coarser than the 'lower limit' topology $\mathcal{T}_\ell$ on $\mathbb{R}$.

(*) 19. Show that for any pair of topologies $\mathcal{T}$ and $\mathcal{T}'$ on $\mathbb{R}$, the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ that they generate **cannot** be equal to the finite complement topology on $\mathbb{R}^2$.

[What does the complement of a box look like?]

20. [Munkres, p.92, Problem #5] If $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on the set X and $\mathcal{O} \subseteq \mathcal{O}^prime$ are topologies on the set Y, show that the product topology $\mathcal{T} \times \mathcal{O}$ on $X \times Y$ is coarser than the topology $\mathcal{T}' \times \mathcal{O}'$. Is the converse result true [i.e., product topology coarser implies that the topologies on each factor are coarser]?