Math 871 Problem Set 1

Starred (*) problems are due Thursday, Sept. 11.

6. For a set X, and (fixed) $a \in X$, the <u>excluded</u> <u>point</u> <u>topology</u> on X is the collection of subsets $\mathcal{T}_a = \{\mathcal{U} \subseteq X \ : \ a \notin \mathcal{U}\} \cup \{X\}$. Show that $\mathcal{T}_a$ is a topology on X.

(*) 7. Show that the set $\mathcal{T} = \{(a,\infty) \ : \ a \in \mathbb{R}\} \ \cup \ \{[a,\infty) \ : \ a \in \mathbb{R}\} \ \cup \ \{\emptyset, \mathbb{R}\}$ of subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$.

[Note that $\mathcal{T}$ can be thought of as the sets $\mathcal{U} \subseteq \mathbb{R}$ so that whenever $a \in \mathcal{U}$ and $b \geq a$, then $b \in \mathcal{U}$.]

8. (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is an increasing function $[a \geq b \ \Rightarrow f(a) \geq f(b)]$, then, as a function from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}, \mathcal{T})$, f is continuous.

(b) Show, conversely, that if $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$ is continuous, then f is increasing!
[Consequently, this topology captures increasing functions precisely as its continuous functions...]
[Hint: suppose f <u>isn't</u> increasing: show it is <u>not</u> continuous.]

(*) 9. [Munkres, p.83, #13.3] (a) Show that for any set X, the sets
$\{\mathcal{U} \subseteq X \ : \ X \setminus \mathcal{U} \text{ is countable}\} \cup \{\emptyset\}$ form a topology on X.

(b) Do the sets $\{\mathcal{U} \subseteq X \ : \ X \setminus \mathcal{U} \text{ is infinite}\} \cup \{\emptyset\}$ always form a topology on X ? Explain why or why not.

10. [Munkres, p.83, #13.4(a)] If $\mathcal{T}_\alpha$ are all topologies on the same set X, show that $\bigcap_\alpha \mathcal{T}_\alpha$ (the intersection of all of the topologies) is also a topology on X. Is $\bigcup_\alpha \mathcal{T}_\alpha$ (their union) a topology on X?

(*) 11. With the topology $\mathcal{T}$ on $\mathbb{R}$ from problem #6, and $\mathcal{T}'$ the "usual" topology on $\mathbb{R}$ (open sets are unions of neighborhoods), show that every continuous function $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}')$ must be <u>constant</u>.

[Hint: Suppose not! Show that f <u>can't</u> be continuous.]

12. With the topologies from problem #11, show that there do exist continuous functions $f : (\mathbb{R}, \mathcal{T}') \to (\mathbb{R}, \mathcal{T})$ which are <u>not</u> constant.

[How could $f^{-1}([a,\infty))$ and $f^{-1}((a,\infty))$ both be open?]