Math 871 Exam 2 Topics Part 2: The homotopy part

Homotopy Theory.

Motivation: understand <u>all</u> continuous functions $f : X \to Y$, since it is functions to/from 'model' spaces that allow us to explore a space.

E.g., paths $= \gamma : I = [0,1] \to X$. How many 'essentially distinct' paths are there from $(-1,0)$ to $(1,0)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$? What is inessetial? <u>Deformations</u>.

Two maps $f, g : X \to Y$ are *homotopic* if one can be deformed to the other (through continuous maps). Formally, there is a cts map $H : X \times I \to Y$ so that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. We write: $f \simeq g$ (via H).

Note: $\gamma_x : t \mapsto H(x,t)$ is a cts path in Y, for every x.

Notation: $f : (X,A) \to (Y,B)$ means $A \subseteq X$, $B \subseteq Y$ and $f(A) \subseteq B$.

Two maps $f, g : (X,A) \to (Y,B)$ are homotopic rel A if $H : X \times I \to Y$ also satisfies $H(a,t) = f(a) = g(a)$ for all $a \in A$, $t \in I$. [So, in part, $f|_A = g|_A$.]

Basic example: any two maps $f, g : X \to \mathbb{R}^n$ are homotopic, via a *straight-line homotopy*: $H(x,t) = (1-t)f(x) + tg(x)$.

Homotopy is an <u>equivalence</u> <u>relation</u>: $f \simeq f$ (via $H(x,t) = f(x)$), $f \simeq g$ implies $g \simeq f$ (via $K(x,t) = H(x, 1-t)$); $f \simeq g$ and $g \simeq h$ implies $f \simeq h$ (via doubling the speed; $M(x,t) = H(x,2t)$ for $t \leq 1/2$ and $= K(x, 2t-1)$ for $t \geq 1/2$).

This allows us to introduce a new notion of equivalence of topological spaces. X and Y are *homotopy equivalent* [we write $X \simeq Y$] if there are $f : X \to Y$ and $g : Y \to X$ so that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$.

Homotopy equivalence is an equivalence relation! Note: a homeomorphism is a homotopy equivalence! [$g \circ f = \mathrm{Id}_X \simeq \mathrm{Id}_X$].

The homotopy viewpoint.

The basic idea is that homotopy equivalence (= 'h.e.') allows us to move past/around 'unimportant' differences in spaces. For example, $\mathbb{R}^2 \setminus \{(0,0)\} \cong S^1 \times \mathbb{R} \simeq S^1 \times I \simeq S^1$ means that maps into $\mathbb{R}^2 \setminus \{(0,0)\}$ 'behave like' maps into S^1 (which we can more readily understand?).

Algebraic topology seeks to understand topological spaces through algebraic invariants. An algebraic invariant assigns to each space X an algebraic object $A(X)$ and to each map $f : X \to Y$ a homomorphism $A(f) : A(X) \to A(Y)$. If X and Y are the 'same', then $A(X)$ and $A(X)$ will be isomorphic. Usually, 'same' means homeomorphic, but we will often find that homotopy equivalent spaces will same the same invariants, due the the methods that we use to build them.

This can be both bad and good, 'homotopy invariance' of a invariant means that it will not be able to distinguish h.e. spaces that are not homeomorphic. But it also means that when computing an algebraic invariant, we can replace a space X with $Y \simeq Y$, which may streamline a computation.

A *retraction* of X onto $A \subseteq X$ is a map $r : X \to A$ so that $r(a) = a$ for all $a \in A$. [A is a *retract* of X]. A is a *deformation retract* of X if $\iota \circ r : X \to A \to X$ is $\simeq \mathrm{Id}_X$ [r is a *deformation retraction*]. and r is a *strong deformation retraction* if $\iota \circ r : (X,A) \to (X,A)$ is $\simeq \mathrm{Id}_X$ rel A (i.e., $H(a,t) = a$ for all $a \in A$). We write $X \searrow A$.

For example, $r : \mathbb{R}^n \searrow \{\vec{0}\}$, since $\iota \circ r \simeq \mathrm{Id}_{\mathbb{R}^n}$ via a straight-line homotopy
$$H(x,t) = (1-t)\iota \circ r(\vec{x}) + t\mathrm{Id}_{\mathbb{R}^n}(\vec{x}) = t\vec{x}\ .$$

A space X is *contractible* if $X \simeq \{*\}$.

Mapping cylinders: If $f : X \to Y$, then $M_f = X \times I \coprod Y/\sim$, where $(x,1) \sim f(x)$. [Idea: we glue $X \times \{1\}$ to Y using f.] Then since $X \times I \searrow X \times \{1\}$, we have $M_f \searrow Y$.

Fact: $f : X \to Y$ is a homotopy equivalence $\Leftrightarrow M_f \searrow X \times \{0\}$. This means that $X \simeq Y \Leftrightarrow$ there is a space Z with $X, Y \subseteq Z$ and $Z \searrow X$, $Z \searrow Y$.

The Fundamental Group.

Idea: find the essentially distinct paths between points in X. How? Turn this into a group! How? The concatenation $\gamma * \eta$ of two paths is a path. But: only if the first ends where the second begins (so that, by the Pasting Lemma, the resulting map is cts). So we either have a partial multiplication (= groupoid!), or we focus on loops $\gamma : (I, \partial I) \to (X, x_0)$ based at a fixed point x_0 9we'll do the second).

Elements of the *fundamental group* $\pi_1(X, x_0)$ 'are' loops; the inverse will be the reverse $\overline{\gamma}(t) = \gamma(1-t)$, since $\gamma * \overline{\gamma} \simeq c_{x_0}$, and the identity element will be the constant map c_{x_0}. But! to make $\gamma * \overline{\gamma}$ equal c_{x_0}, we need to work with *homotopy classes* of loops. So elements really are equivalence classes $[\gamma]$ of loops, under $\simeq$ rel ∂I.

Then by building homotopies (mostly working on the domain I, i.e., building $K = \gamma \circ H : I \times I \to I \to X$) we can see that $[\gamma][\eta] = [\gamma * \eta]$ is well defined, $[\gamma]^{-1} = [\overline{\gamma}]$ is the inverse, and $([\gamma][\eta])[\omega] = [\gamma]([\eta][\omega])$, so under $*$, $\pi_1(X, x_0)$ is a group. [Most of the proofs that needed maps (like $(\gamma * \eta) * \omega$ and $\gamma * (\eta * \omega)$ (which are the same concatenations, except at 4,4, and 2 times speed, versus 2,4, and 4 times speed) are homotopic can be given 'picture' proofs, in addition to explicit analytic formulas.

Given a map $f : (X, x_0) \to (Y, y_0)$, we get an induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ via $f_*[\gamma] = [f \circ \gamma]$. This is well-defined, and a homomorphism.

Basic computations: $\pi_1(\{*\}, *) = \{1\}$, as are $\pi_1(\mathbb{R}^n, \vec{0})$ and $\pi_1([0,1]^n, x_0)$ for any x_0. More generally, any contractible space has trivial fundamental group.

Since $(f \circ g)_* = f_* \circ g_*$, and $(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X, x_0)}$, then $X \cong Y$ via f implies $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

More generally, if $f : X \to Y$ is a h.e., then f_* is an isomorphism, but, because of basepoint issues, the inverse of f_* is generally not g_* for g a homotopy inverse. Th is is because under a homotopy $H : X \times I \to X$ of $g \circ f$ to $\mathrm{I}d$, the basepoint x_0 traces out a path η from $g(f(x_0)) = x_1$ to x_0, and $[g \circ f \circ \gamma] = [\overline{\eta} * \gamma * \eta]$. This map $[\gamma] \mapsto [\overline{\eta} * \gamma * \eta]$ from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ is a *change of basepoint isomorphism*, which we might call η_* ? The fact that homotopies can drag basepoints around will be a theme we will return to many times moving forward.

If X is path connected, then, up to isomorphism, $\pi_1(X, x_0)$ is independent of x_0 (we can always find a path to effect an isomorphism), and so we will often write $pi_1(X)$, when X is path-connected, when we only care about the abstract group.

$\pi_1(S^1, (1,0)) \cong \mathbb{Z}$. The main ingredients:

Writing $S^1 \subseteq \mathbb{C}$ and $\gamma_n(t) = e^{2\pi int}$ is the loop traversing S^1 n times counterclockwise at uniform speed, then (1) every loop γ at $(1,0)$ is $\simeq \gamma_n$ for some n.

We define $w : \pi_1(S^1,(1,0)) \to \mathbb{Z}$ by $w[\gamma] = n$ if $[\gamma] = [\gamma_n]$. This is well-defined: (2) if $\gamma_n \simeq \gamma_m$ rel endpoints, then $n = m$.

w is a bijective homomorphism!

The proof of (1) amounted to making a general γ progreessively nicer, via homotopy. This involved

Lebesgue Number Theorem: If (X,d) is a compact metric space and $\{U_\alpha\}$ is an open covering of X, then there is an $\epsilon > 0$ so that for every $x \in X$ there is an $\alpha = \alpha(x)$ so that we have $N_d(x,\epsilon) \subseteq U_\alpha$.

Then by covering S^1 by the 'top 2/3rds' and 'bottom 2/3rds' subsets and taking inverse images under $\gamma : (I,\partial I) \to (S^1,(1,0))$, the LNT will partition I into finitely many intervals each mapping into top or bottom. Creating subpaths by restricting to each subinterval, and inserting 'hairs' to points $(1,0),(-1,0)$ in the intersection of top and bottom, we can then homotope the subpaths to standard paths $t \mapsto e^{\pm 2\pi it}$. Cancelling pairs the reverse direction give us our 'normal forms' γ_n.

The proof of (2) amounted to using an 'extra' coordinate $(\cos t, \sin t, t)$ to keep track of how many times we wind around the circle. To do this correctly, we really use the map $p : t \mapsto (\cos t, \sin t, t) \mapsto (\cos t, \sin t)$ and then <u>lift</u> paths $\gamma : I \to S^1$ to paths $\widetilde{\gamma} : I \to \mathbb{R}$ with $\gamma = p \circ \widetilde{\gamma}$. This agin uses the LNT to partition I into subintervals mapping into top and bottom, and the fact that the inverse image of top and bottom are a disjoint union of open sets mapped homeomorphically under p to the top and bottom. [This is the *evenly covered property*.]

More than this, homotopies $H : I \times I \to S^1$ can also be lifted; this enables us to show that loops homotopic rel endpoints, when lifted both starting at the same point, will end at the same point. Since γ_n when lifted starting at 0 will end at n, the result follows.

Applications. This single computation has many applications! First, there is no retraction $r : \mathbb{D}^2 \to \partial\mathbb{D}^2$. This is because if there were one, then $r_* : \pi_1(\mathbb{D}^2,(1,0)) \to \pi_1(S^1,(1,0))$ would be a surjection, which is impossible.

This in turn gives the *Brouwer Fixed Point Theorem*: Every countinuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point. For if not, we can then manufacture a retraction $r : \mathbb{D}^2 \to \partial\mathbb{D}^2$.

Finally, we can prove the *Fundamental Theorem of Algebra*: Every non-constant polynomial p has a complex root. For if not, then for large enough N the map

$$t \mapsto f(Ne^{2\pi it}) \mapsto f(Ne^{2\pi it})/||f(Ne^{2\pi it})||$$

from I to S^1 is homotopic to both $c_{(1,0)} = \gamma_0$ and γ_n for $n =$ the degree of f, a contradiction.