Math 871 Exam 1 Topics

Starting point: continuity

$f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if $|f(x) - f(x_0)|$ is small so long as $|x - x_0|$ is small <u>enough</u>.

Goal: make this notion make sense more generally, and determine what makes the fundamental results (extreme value theorem, intermediate value theorem) work.

Two ideas help: a neighborhood of x is one which contains all points 'close enough' to x. The *inverse image* of a set is the collection of points that the function f maps into x. Then: continuity at x means that the inverse image of a neighborhood of $f(x)$ is a neighborhood of x.

Images and inverse images.

Given $f : X \to Y$ a function, and $A \subseteq X$, $B \subseteq Y$, we have images $f(A) = \{f(x) \ : \ x \in A\}$ and inverse images $f^{-1}(B) = \{x \in X \ : \ f(x) \in B\}$

Inverse images are very well-behaved! $f^{-1}(\cap_\alpha B_\alpha) = \cap_\alpha f^{-1}(B_\alpha)$, $f^{-1}(\cup_\alpha B_\alpha) = \cup_\alpha f^{-1}(B_\alpha)$, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.

But images are not as well-behaved: $f(\cup_\alpha A_\alpha) = \cup_\alpha f(A_\alpha)$, but only $f(\cap_\alpha A_\alpha) \subseteq \cap_\alpha f(A_\alpha)$ in general.

Finite/countable/uncountable sets.

A set S is finite if for some $n \in \mathbb{Z}_+$ there is a bijective function $S \leftrightarrow \{1, \ldots, n\}$

Equivalently, for some n, $\{1, \ldots, n\} \twoheadrightarrow S$, or $S \hookrightarrow \{1, \ldots, n\}$.

Some results: if A, B are finite, then $A \cup B$, $A \times B$, and the set of all functions $\{f : A \to B\}$ are finite.

If B is finite and $A \subseteq B$ then A is finite.

Countably infinite: there is a bijection $S \leftrightarrow \mathbb{Z}_+$.

Countable: finite or countable infinite. Equivalently, there exists a surjection $\mathbb{Z}_+ \twoheadrightarrow S$, or there exists an injection $S \hookrightarrow \mathbb{Z}_+$.

Infinite: not finite! Equivalently, there is an injection $\mathbb{Z}_+ \hookrightarrow S$ (or a surjection $S \twoheadrightarrow \mathbb{Z}_+$)

Uncountable: not countable! Equivalently, there is <u>no</u> surjection $\mathbb{Z}_+ \twoheadrightarrow S$ (or <u>no</u> injection $S \hookrightarrow \mathbb{Z}_+$).

Examples: $\mathbb{R}$, or all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ (note: the second is equivalent to all subsets of $\mathbb{Z}_+$, via $f \leftrightarrow f^{-1}(\{0\})$)

Back to continuity.

Given a *metric* on a set X

[a function $d : X \times X \to \mathbb{R}$ with, for all $x, y, z \in X$, (1) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$, (2) $d(x, y) = d(y, x)$, and (3) $d(x, z) \leq d(x, y) + d(y, z)$]

We can formalize continuity at $x_0 \in X$ using neighborhoods $N_d(x_0, \epsilon) = \{x \in X \ : \ d(x_0, x) < \epsilon\}$; we need $f^{-1}(N_{d'}(f(x_0), \epsilon))$ contains some $N_d(x_0, \delta)$. But more; since $x \in N_d(x_0, \epsilon)$ implies that $N_d(x, \epsilon - d(x, x_0)) \subseteq N_d(x_0, \epsilon)$, we have that $x \in f^{-1}(N_{d'}(f(x_0), \epsilon))$ implies that $N_d(x, \delta) \subseteq f^{-1}(N_{d'}(f(x_0), \epsilon))$ for some $\delta > 0$. That is, $f^{-1}(N_{d'}(f(x_0), \epsilon))$ is, for every $x_0 \in X$, a union of neighborhoods.

This leads to: $U \subseteq X$ is *open* if it is a union of neighborhoods. And $f : X \to Y$ is continuous if $f^{-1}(V)$ is open (in X) for <u>every</u> $V \subseteq Y$ open (in Y). And we just eliminated (explicit) mention of a point; continuity 'at' x_0 has been eliminated, leaving just 'continuity'.

Topologies.

Looking at what properties open sets in a metric space have leads us to the notion of a topology on a set X: it is a collection $\mathcal{T}$ of subsets of X so that

(1) $\emptyset, X \in \mathcal{T}$

(2) if $U_\alpha \in \mathcal{T}$ for all α, then $\cup_\alpha U_\alpha \in \mathcal{T}$

(3) if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$

And a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ between *topological spaces* (i.e., sets with particular topologies) is *continuous* (cts) if $f^{-1}(V) \in \mathcal{T}$ for every $V \in \mathcal{T}'$.

Constant maps are always cts; compositions of cts maps are cts.

Examples.

X = any set, $\mathcal{T} = \mathcal{P}(X)$ - all subsets of X, the *discrete* topology.

X = any set, $\mathcal{T} = \{\emptyset, X\}$, the *indiscrete* (or trivial) topology.

X = any set, $\mathcal{T}_f = \{A \subseteq X \ : \ X \setminus A \text{ is finite}\} \cup \{\emptyset\}$, the *finite complement* topology.

X = any set, $\mathcal{T}_c = \{A \subseteq X \ : \ X \setminus A \text{ is countable}\} \cup \{\emptyset\}$, the *countable complement* topology.

X = any set, $a \in X$, $\mathcal{T}_{ip} = \{A \subseteq X \ : \ a \in A\} \cup \{\emptyset\}$, the *included point* topology.

X = any set, $a \in X$, $\mathcal{T}_{ip} = \{A \subseteq X \ : \ a \notin A\} \cup \{X\}$, the *excluded point* topology.

$X = \mathbb{R}$, $\mathcal{T}_{irr} = \{(a\infty)\} \cup \{\emptyset, \mathbb{R}\}$, the *infinite ray* (to the right) topology.

X = any metric space,$\mathcal{T}_d = \{\text{all unions of } d\text{-neighborhoods in } X\} = \{\cup_i N_d(x_i, \epsilon_i) \ : \ x_i \in X, \epsilon_i > 0\}$, the *metric* topology.

$\mathcal{T}, \mathcal{T}'$ topologies on X, with $\mathcal{T} \subseteq \mathcal{T}'$, we say $\mathcal{T}$ is coarser/smaller than $\mathcal{T}'$, and $\mathcal{T}'$ is finer/larger than $\mathcal{T}$.

Basic idea: coarser $\Rightarrow$ more cts functions into X (fewer inverse image to check), finer $\Rightarrow$ more cts functions out of X (more likely to contain the inverse images).

$f : (X\mathcal{T}) \to Y$. $\mathcal{T}' = \{V \subseteq Y \ : \ f^{-1}(V) \in \mathcal{T}\}$ is the finest topology on Y making f cts.

$g : X \to (Y, \mathcal{T}')$. $\mathcal{T} = \{f^{-1}(V) \ : \ V \in \mathcal{T}'\}$ is the coarsest topology on X making f cts.

A set $U \subseteq X$ is open $\Leftrightarrow$ for every $x \in U$ there is $U_x \in \mathcal{T}$ with $x \in U_x \subseteq U$. [Note: if you know U is open, $U_x = U$ works!]

Bases/subbases.

Metric topologies are defined as unions of neighborhoods. What makes this a topology?

A basis $\mathcal{B}$ for a topology on X is a collection of subsets so that

(1) union of elements of $\mathcal{B}$ is X [every $x \in X$ lies in some $B \in \mathcal{B}$]

(2) the intersection of two is a union of elements of $\mathcal{B}$ [if $B, B' \in \mathcal{B}$ and $x \in B \cap B'$ then $x \in B'' \subseteq B \cap B'$ for some $B'' \in \mathcal{B}$]

$\mathcal{T}(\mathcal{B})$ = unions of elements of $\mathcal{B}$ = the topology generated by $\mathcal{B}$, the coarsest toplology containing $\mathcal{B}$

$f : (X, \mathcal{T}) \to (Y, \mathcal{T}(\mathcal{B}))$ is cts $\Leftrightarrow f^{-1}(B) \in \mathcal{T}$ for all $B \in \mathcal{B}$

Examples:

$\mathcal{B} = \{(a, b) \subseteq \mathbb{R} \ : \ a, b \in \mathbb{R}\}$ is a basis for the usual topology on $\mathbb{R}$; restrict to $a, b \in \mathbb{Q}$, still a basis for usual topology.

$\mathcal{B} = \{[a, b) \subseteq \mathbb{R} \ : \ a, b \in \mathbb{R}\}$ is a basis; $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B})$ = the *lower limit* topology on $\mathbb{R}$. Strictly finer than the usual topology!

Subbasis $\mathcal{S}$: insist only on (1) union of elements of $\mathcal{S}$ is X.

$\mathcal{B} = \mathcal{B}(\mathcal{S}) = \{S_1 \cap \cdots \cap S_n \ : \ n \in \mathbb{Z}_+, S_i \in \mathcal{S}\}$ is a basis; $\mathcal{T}(\mathcal{S}) = \mathcal{T}(\mathcal{B})$ = topology generated by $\mathcal{S}$.
Detecting bases: If $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{T}(\mathcal{B})$, then $\mathcal{T} = \mathcal{T}(\mathcal{B})$

Product topologies.
$(X, \mathcal{T})$, $(Y, \mathcal{T}')$ top spaces, then $\mathcal{B} = \mathcal{T} \times \mathcal{T}'$ (subsets of $X \times Y$) is not a topology, but it is a basis for one (closed under intersection). $\mathcal{T}(\mathcal{T} \times \mathcal{T}')$ = (abusing notation '$\mathcal{T} \times \mathcal{T}'$' is the *product topology* on $X \times Y$.
Motivation: want projection maps $p_X : X \times Y \to X$, $p_X(x, y) = x$, etc. continuous! This forces topology on $X \times Y$ to contain certain sets, coarsest topology that contains them is product topology.
If $\mathcal{T} = \mathcal{T}(\mathcal{B})$, $\mathcal{T}' = \mathcal{T}(\mathcal{B}')$, then $\mathcal{B} \times \mathcal{B}'$ is a basis for the product topology.
Example: product topology on $\mathbb{R}^2$, $\mathbb{R}^n$. Same as the (usual) metric topologies!
Arbitrary products: two choices! $\prod_i X_i$, coarsest topology making all projections cts is the *product topology*, it has subbasis $\{p_{X_i}^{-1}(U) \ : \ i \in I, u \in \mathcal{T}_i\}$ (basis is $\prod_i U_i$, where all but finitely many U_i are X_i).
Box topology: basis is $\{\prod_i U_i \ : \ U_i \in \mathcal{T}_i\}$; all factors can be proper open sets.
If I is infinite, box topology is (generally) strictly finer than product topology
Recognizing cts fcns: $f : (X, \mathcal{T}) \to (\prod_i X_i, \text{prod top})$ is cts $\Leftrightarrow p_{X_i} \circ f : X \to X_i$ is cts for all i.
This is not true if $\prod_i X_i$ is given the box topology!

Subspaces.
Motivation $(X, \mathcal{T})$ and $A \subseteq X$, would be useful if inclusion map $\iota : A \hookrightarrow X$ is cts.
$\mathcal{T}_A = \{A \cap U \ : \ U \in \mathcal{T}\}$ is a topology on A, the coarsest making ι cts ($A \cap U = \iota^{-1}(U)$), called the *subspace topology* on A.
$f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ cts and $A \subseteq X$, then $f \circ \iota = f|_A : (A, \mathcal{T}_A) \to (Y, \mathcal{T})$ is cts
$\mathcal{T} = \mathcal{T}(\mathcal{B})$, then $\mathcal{B}_A = \{A \cap B \ : \ B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_A$
$A \subseteq X$, $B \subseteq Y$, then $A \times B \subseteq X \times Y$, and the subspace topology on $A \times B$ is the same as the product of the subspace topologies (same bases!)
Subtlety: $f|_A : A \to Y$ cts means $A \cap f^{-1}(U)$ is open in A, i.e., $A \cap f^{-1}(U) = A \cap V$ for some V open in X. [I.e., $f|_A$ can be cts when f isn't!]

Closed sets.
$(X, \mathcal{T})$ top space, $C \subseteq X$ is *closed* (w.r.t. $\mathcal{T}$; omitted if clear from context) if $X \setminus C \in \mathcal{T}$. [Equivalently, $C = X \setminus U$ for some $U \in \mathcal{T}$]
C_i closed $\Rightarrow \cap_i C_i$ closed; $C, D \subseteq X$ closed $\Rightarrow C \cap D$ closed.
One can cast everything about a topology in terms of closed sets; e.g., a fcn $f : X \to Y$ is cts $\Leftrightarrow f^{-1}(C) \subseteq X$ is closed for evvery $c \subseteq Y$ closed.
$C \subseteq X$, $D \subseteq Y$ both closed $\Rightarrow C \times D \subseteq X \times Y$ is closed. ($\Leftarrow$ requires both sets non-empty)

Closure.
$C \subseteq A \subseteq X$ is closed in $(A, \mathcal{T}_A) \Leftrightarrow C = A \cap D$ for $D \subseteq X$ closed.
Closed sets are closed (no put intended) under intersection, so can (usually) find a smallest closed set satisfying a property, as $\cap$\{closed sets with ppty\}
Open set closed under union, so can find largest open set with a property.
Example: $A \subseteq X$, the closure of A in $X = \overline{A} = \text{cl}_{\mathcal{T}}(A) = \cap\{C \subseteq X \text{ closed} \ : \ A \subseteq C\}$ = smallest closed set containing A

Interior of $A = \text{int}(A) = \text{int}_{\mathcal{T}}(A) = \cap\{U \subseteq X \ : \ U \in \mathcal{T}, U \subseteq A\}$

$x \in \overline{A} \Leftrightarrow x \in C$ for every closed C with $A \subseteq C \Leftrightarrow$ whenever $U \in \mathcal{T}$ and $x \in U$ we have $A \cap U \neq \emptyset$

x is a *limit point* of A if whenever $U \in \mathcal{T}$ and $x \in U$ we have $(A \setminus \{x\}) \cap U \neq \emptyset$, i.e., $x \in \overline{A \setminus \{x\}}$.

The set of limit points of A is denoted A'. So $\overline{A} = A \cup A'$; A is closed $\Leftrightarrow A' \subseteq A$

If $B \subseteq A \subseteq X$, then $\text{cl}_A(B) = A \cap \text{cl}_X(B)$. If $A \subseteq X$ and $B \subseteq Y$, then $\text{cl}_{X \times Y}(A \times B) = \text{cl}_X(A) \times \text{cl}_Y(B)$.

For any $A \subseteq X$, $\text{cl}(\text{int}(\text{cl}(\text{int}(A)))) = \text{cl}(\text{int}(A))$. This is the main ingredient in the "14 Set Theorem": at most 14 distinct set can be constructed from A using a combination of closure and complement. [$X \setminus \text{cl}(X \setminus A) = \text{int}(A)$]

For $f : X \to Y$ cts, $\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A})$, and f is cts $\Leftrightarrow f(\overline{A}) \subseteq \overline{f(A)}$ for every subset $A \subseteq X$.

Building continuous functions.

$f : X \to Y$, $X = \cup_i U_i$ with $U_i \in \mathcal{T}$ for all i, then f is cts $\Leftrightarrow f|_{U_i}$ is cts for all i.

Reverse: (Pasting Lemma) If $X = \cup_i U_i$ with U_i open, and $f_i : U_i \to Y$ are cts for all i, and $f_i = f_j$ on $U_i \cap U_j$ for all i, j, then $f : X \to Y$ defined by $f(x) = f_i(x)$ if $x \in U_i$ is (well-defined and) cts.

Closed set version: If $f : X \to Y$ $X = C_1 \cup \cdots \cup C_n$ with C_i closed for all i, then f is cts $\Leftrightarrow f|_{C_i}$ is cts for all i.

Reverse: ('Other' Pasting Lemma) If $X = C_1 \cup \cdots \cup C_n$ with C_i closed for all i, $f_i : C_i \to Y$ are cts for all i, and $f_i = f_j$ on $C_i \cap C_j$ for all i, j, then $f : X \to Y$ defined by $f(x) = f_i(x)$ if $x \in C_i$ is (well-defined and) cts.

Homeomorphisms.

For fcns on $\mathbb{R}$, we have the Inverse Function Theorem: a cts bijection $f : \mathbb{R} \to \mathbb{R}$ has contiuous inverse. But for topological spaces, this does not hold!

Example: $X = \mathbb{Z}$, $\mathcal{T} = \{A \subseteq \mathbb{Z} \ : \ A \subseteq \mathbb{Z}_+ \text{ or } A = \mathbb{Z}\}$, then $f : X \to X$ given by $f(x) = x - 1$ is a cts bijection, but $f^{-1}(x) = x + 1$ is not cts.

A *homeomorphism* is a cts bijection $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that the inverse $g = f^{-1} : (Y, \mathcal{T}') \to (X, \mathcal{T})$ is also cts. I.e., f is a bijection so that $V \in \mathcal{T}' \Leftrightarrow f^{-1}(V) \in \mathcal{T}$. We write $X \cong Y$ (if the topologies are understood).

Examples: $\mathbb{R} \cong (0, 1) \cong (a, b) \cong (a, \infty) \cong (-infty, a)$. $[0, 1] \cong [a, b]$

A homeo therefore gives a bijection not only of the points of X and Y, but also of their open sets (via inverse images). Consequently, any property that can be expressed in terms of points and open (or closed) sets which is true for one of X and Y must be true for the other. Such properties are called *topological properties*.

A topological property is one that is preserved by homeomorphisms.

Examples.

$(X, \mathcal{T})$ is T_1 if every 1-point set $\{x\}$ is closed.

$(X, \mathcal{T})$ is T_2 or *Hausdorff* (Hdf?) if for every pair of points $x, yinX$, if $x \neq y$ then there are $U, V \in \mathcal{T}$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$. [Points can be separated using disjoint open sets.]

T_2 implies T_1; Metric topologies are Hausdorff.

$(X, \mathcal{T})$ is *path connected* if for every $x, y \in X$ there is a path, a cts fcn $\gamma : ([0, 1], \text{usual}) \to (X, \mathcal{T})$ so that $\gamma(0) = x$ and $\gamma(1) = y$.

A set $A \subseteq X$ is *dense* if $\overline{A} = X$. A space X is *separable* if it contains a countable dense subset. [For example, $(\mathbb{R}, \text{usual})$ is separable; $\mathbb{Q}$ is dense.]

A space is *second countable* if its topology can be generated by a basis consisting of countably many sets. [For example, $(\mathbb{R}, \text{usual})$ is 2nd ctble.]

At its heart, topology is the study of topological properties!, and the relationships between them.

Restriction of range: if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous, and $f(X) \subseteq B \subseteq Y$, then f, thought of as a function $f :: (X, \mathcal{T}) \to (B, \mathcal{T}'_B)$, is also continuous.

A *topological embedding* (or imbedding) is an injective cts map $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ so that, restricting the range, $f : (X, \mathcal{T}) \to (f(X), \mathcal{T}'_{f(X)})$ is a homeomorphism.

Quotient spaces.

We have seen, for $f : (X, \mathcal{T}) \to Y$, there is a finest topology on Y to make it cts: $\mathcal{T}' = \{V \subseteq Y \ : \ f^{-1}(V) \in \mathcal{T}\}$. This topology is especially important/useful when f is surjective:

A *quotient map* is a surjective fcn $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ with $V \in \mathcal{T}' \Leftrightarrow f^{-1}(V) \in \mathcal{T}$.

$U = f^{-1}(V)$ is called a *saturated* set; U contains every inverse impage that it meets. A surjective f is a quotient map $\Leftrightarrow$ it is continuous and the image of every saturated open set in X is open in Y [the point: f surjective means that $f(f^{-1}(V)) = V$].

Using closed sets (i.e., complements), quotient maps are also the surjective cts maps for which the image of every saturated closed st is closed.

So for example, a cts, open (the image of <u>every</u> open set is open) surjective function is a quotient map. And a cts closed (the iage of <u>every</u> closed set is closed) surjective function is a quotient map.

So the projection maps $p_{X_j} : \prod_i X_i \to X_j$ are quotient maps (they are cts open surjections).

Given a top space $(X, \mathcal{T})$, an equivalence relation $\sim$ on X [reflexive, symmetric, transitive] has a collection $X/\sim = Y$ of equivalence classes $[x] = \{y \in X \ : \ x \sim y\}$, and a surjective function $q : X \to Y = X/\sim$. Giving Y the topology $\mathcal{T}' = \{V \subseteq Y \ : \ q^{-1}(V) \in \mathcal{T}\}$ makes q a quotient map; we call $\mathcal{T}'$ the *quotient topology* on Y (induced from q). Viewing $\sim$ as describing how to glue pieces of X together, this construction is a 'standard' way to build topological spaces.

E.g., on $X = [0, 1]$, the equivalence relation $x \sin y$ if $x = y$ or $\{x, y\} = \{0, 1\}$ 'glues' the ends of the interval together, and (we shall see!) yields a quotient space homeomorphic to the unit circle in $\mathbb{R}^2$.

Building cts fcns, II: If $q : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a quotient map and $f : (X, \mathcal{T}) \to (Z, \mathcal{T}'')$ is cts, satisfying $q(x) = q(y)$ implies $f(x) = f(y)$, then there is an induced map $\overline{f} : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$, defined by $\overline{f}([x]) = f(x)$, and $\overline{f}$ is continuous.

So, for example, the map $f : [0, 1] \to S^1 \subseteq \mathbb{R}^2$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$, respects the equivalence relation above, and so induces a cts (bijection) $\overline{f} : [0, 1]/\sim \to S^1$. (We will eventually see that this is a homeo!)

Most of our 'interesting' topological spaces will be built as quotient spaces, and we will recognize what they are by building maps along the lines above.

BUT: quotient maps do not behave well under most of our constructions. The composition of two quotient maps <u>is</u> a quotient map. But the restriction of a quotient map to a subspace (restricting the domain, too) will not, in general, be a quotient map. And if $f : (X_1, \mathcal{T}_1) \to (Y_1, \mathcal{T}'_1)$ and $g : (X_2, \mathcal{T}_2) \to (Y_2, \mathcal{T}'_2)$ are both quotient maps, the map$f \times g : X_1 \times X_2 \to Y_1 \times Y_2$

need not be a quotient map (a situation that will bother us many times moving forward...). But if f and g are both open maps, then $f \times g$ is an open map, and so is a quotient map.

Connectedness.

What makes the Intermediate Value Theorem [$f : [a, b] \to \mathbb{R}$ cts, then $f([a, b])$ contains the interval between $f(a)$ and $f(b)$] work? What is it about the topological space $[a, b]$? Pretend that the result fails! $f : X \to \mathbb{R}$

Then there is a c between $f(a)$ and $f(b)$ missed by f, so $f(X) \subseteq (-\infty, c) \cup (c, \infty)$, so $X = f^{-1}(-\infty, c) \cup f^{-1}(c, \infty) = U \cup V$, with $U, V \in \mathcal{T}$, $U \cap V = f^{-1}(\emptyset) = \emptyset$, and $U, V \neq \emptyset$ (a is in one, b in the other).

This describes a topological property! A *separation* of a top space $(X, \mathcal{T})$ is a pair $U, V \in \mathcal{T}$ with $U \cup V = X$, $u \cap V = \emptyset$, and $U, V \neq \emptyset$. And denying a topological property is a topological property! A space is *connected* if it has <u>no</u> separation.

Equivalently, if $U, V \in \mathcal{T}$ with $U \cup V = X$ and $U \cap V = \emptyset$, then either $U = \emptyset$ or $V = \emptyset$. [Equivalently, $U = X$ or $V = X$!]

Equivalently (since $U = X \setminus V$ would be closed), X connected means that X contains no non-trivial (not equal to $\emptyset, X$) <u>clopen</u> (= both closed and open) sets.

A separation can be used to build a cts fcn $f : X \to \mathbb{R}$ failing IVT ($f(x) = 3$ if $x \in U$ and $f(x) = 71$ if $x \in V$ works...).

So $(X, \mathcal{T})$ is connected $\Leftrightarrow$ for <u>every</u> cts function $f : (X, \mathcal{T}) \to (\mathbb{R}, \text{usual})$ and $a, b \in X$, if c lies between $f(a)$ and $f(b)$ then there is a $d \in X$ with $f(d) = c$.

A subset $A \subseteq X$ is connected if $(A, \mathcal{T}_A)$ is a connected space. Equivalently, if $A \subseteq U \cup V$ with $U, V \in \mathcal{T}$ and $A \cap U \cap V = \emptyset$, then either $A \subseteq U$ or $A \subseteq V$.

The unit interval ($[0, 1]$usual) is connected.

If $f : x \to Y$ is cts and X is connected, then $f(X) \subseteq Y$ is a conected subset of Y. ['The cts image of a connected set is connected.'] (Otherwise, the inverse image of a separation of $f(X)$ will be a separation of X.)

Flipside: a space is *totally disconnected* if for every $x, y \in X$ with $x \neq y$, there is a separation (U, v) of X with $x \in U$, $y \in V$. [Points can be separated using separations!]

'Building' connected sets.

If $(X, \mathcal{T})$ is path connected, then $(X, \mathcal{T})$ is connected. (Otherwise, a path between points in different sets of the separation will, taking inverse images, give a separation of $[0, 1]$.

But: connected spaces need not be path connected. (Example: any uncountable set X with the countable complement topology. Every cts fcn $\gamma : [0, 1] \to X$ must be constant.)

All intervals in $\mathbb{R}$ are path-connected, hence connected. Conversely, all connected subsets of $\mathbb{R}$ are intervals.

If $A, B \subseteq X$ are connected subsets of X, and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

If $A_i \subseteq X$ are connected subsets, and for some j, $A_i \cap A_j \neq \emptyset$ for all i, then $\cup_i A_i$ is connected.

If $A \subseteq X$ is connected, and $A \subseteq B \subseteq \overline{A}$, then B is connected.

If X and Y are connected, then $X \times Y$ is connected.

If X_i are all connected, then $\prod_i X_i$, using the <u>product</u> topology, is connected.

But: $\prod_{i \in \mathbb{Z}_+} \mathbb{R}$, with the <u>box</u> topology, is not connected. (Thinking of points as sequences, the sets {bounded sequences}, {unbounded sequences} form a separation).