Math 856 Problem Set 2

Starred (*) problems to be handed in Thursday, October 6

(*) 8. [Lee, problem 2-17] Find an example of a (non-closed: it can't be done if the set is closed!) subset A of a smooth manifold M, and a smooth function $f : A \to \mathbb{R}$ which admits **no** extension to a smooth function $\tilde{f} : M \to \mathbb{R}$.

(Recall that f is called smooth if for every $x \in A$ there is a neighborhood $x \in \mathcal{U}$ and a smooth extension of $f|_{A\cap\mathcal{U}}$ to the neighborhood $\mathcal{U}$.) [Hint: this is really about no <u>continuous</u> extension...]

(*) 9. Show that a function $f : M^n \to N^m$ is $C^\infty \Leftrightarrow g \circ f : M^n \to \mathbb{R}$ is C^∞ for *every* C^∞ function $g : N^m \to \mathbb{R}$. [Hint: you might find bump functions useful.]

10. We know that if $C, D \subseteq M$ are disjoint closed sets of the smooth manifold M, then there exists a smooth function $f : M \to [0,1]$ with $C \subseteq f^{-1}(0)$ and $D \subseteq f^{-1}(1)$. But we can if fact make these containments *equalities*:

(a) Show that it suffices to build a smooth function $g : M \to [0,1]$ with $C = g^{-1}(0)$.

(b) Build a countable cover $\{U_i\}$ of $M \setminus C$ by open sets of the form $h_i^{-1}(B(x_i,1)$ for a collection of coordinate charts h_i. Build C^∞ functions $g_i : M \to \mathbb{R}$ which are > 0 in U_i and $= 0$ on $M \setminus U_i$. Note that $\overline{U_i}$ is compact; for each i, let

$$\alpha_i = \sup_{x\in\overline{U_i};j\leq i;k_1,\ldots k_i\leq n}\{\partial^k g_j/\partial x_{k_1}\cdots\partial x_{k_i}(x)\} .$$

Show that the function $g = \sum g_i/(\alpha_i 2^i)$ is C^∞ and $C = g^{-1}(0)$.

11. Giving $M_1 \times M_2$ the product smooth structure, show that $f : N \to M_1 \times M_2$ is smooth $\Leftrightarrow$ the maps $p_1 \circ f : N \to M_1$, $p_2 \circ f : N \to M_2$ are smooth, where p_1, p_2 are the projections onto the first and second factors, respectively. Show, moreover, that the product smooth structure is the only smooth structure on $M_1 \times M_2$ with this property.

12. For $a \in M$, let $\mathcal{F}_a \subseteq C^\infty(M)$ denote the smooth functions satisfying $f(a) = 0.$ and let $L : \mathcal{F}_a \to \mathbb{R}$ be a linear operator satisfying $L(fg) = 0$ for all $f, g \in \mathcal{F}_a$. Show that there is a unique derivation $X \in T_aM$ satisfying $X|_{\mathcal{F}_a} = L$.

(N.B. This provides still another characterization of tangent vectors, has the vector space of linear maps $X : \mathcal{F}_a/W \to \mathbb{R}$, where $W = \mathcal{F}_a^2 =$ the ideal generated by products fg for $f, g \in \mathcal{F}_a$.)

(*) 13. Using our 'standard' charts on the sphere S^n (domains are the $2n$ open hemispheres, and maps ignore the coordinate whose sign is being restricted), show that the map $F : S^3 \to S^2$ given by

$$F(x, y, z, w) = (2xz + 2yw, 2yz - 2xw, x^2 + y^2 - z^2 - w^2)$$

is smooth (here we write points of S^3 as points of $\mathbb{R}^4$ satisfying $x^2+y^2+z^2+w^2 = 1$). [Check also that it maps into S^2 !]

[N.B.: This is a fairly famous map, known as the *Hopf map*. Writing points in $\mathbb{R}^4$ as pairs of complex numbers (z_1, z_2), this map can be expressed as $F(z_1, z_2) = (2z_1\overline{z_2}, |z_1|^2 - |z_2|^2)$. (This was stolen straight from Wikipedia.) If $F(z_1, z_2) = F(w_1, w_2)$, then $z_1 = \lambda w_1$ and $z_2 = \lambda w_2$ for some complex number λ with $|\lambda| = 1$. This implies that every point inverse is a 'round' circle.]