Math 856 Homework 2

Starred (*) problems to be handed in Thursday, September 21

(*) 9: Let $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. Suppose that $f : \mathbb{H}^n \to \mathbb{R}$ is a function for which every $x \in \mathbb{H}^n$ has an open neighborhood U_x of x such that f extends to a C^∞ function on U_x. Show that f extends to a C^∞ function on a neighborhood of $\mathbb{H}^n$ in $\mathbb{R}^n$. (Short version: show that if f is locally smooth, then f is smooth. A well-chosen partition of unity might help?)

10: We know that if $C, D \subseteq M$ are disjoint closed sets of the smooth manifold M, then there exists a smooth function $f : M \to [0,1]$ with $C \subseteq f^{-1}(0)$ and $D \subseteq f^{-1}(1)$. But we can if fact make these containments *equalities*:

(a) Show that it suffices to build a smooth function $g : M \to [0,1]$ with $C = g^{-1}(0)$.

(b) Build a countable cover $\{U_i\}$ of $M \setminus C$ by open sets of the form $h_i^{-1}(B(x_i, 1)$ for a collection of coordinate charts $h_i = (x^1, \dots, x^n)$ with image containing $B(x_i, 2)$. Build C^∞ functions $g_i : M \to \mathbb{R}$ which are > 0 in U_i and $= 0$ on $M \setminus U_i$. Note that $\overline{U_i}$ is compact; for each i, let

$$\alpha_i = \sup_{x \in \overline{U_i}; j \leq i; m \leq i; k_1, \dots k_m \leq n} \{\frac{\partial^m g_j}{\partial x^{k_1} \cdots \partial x^{k_m}}(x)\} .$$

Show that the function $g = \sum g_i/(\alpha_i 2^i)$ is C^∞ and $C = g^{-1}(0)$.

11: Giving $M_1 \times M_2$ the product smooth structure, show that $f : N \to M_1 \times M_2$ is smooth $\Leftrightarrow$ the maps $p_1 \circ f : N \to M_1$, $p_2 \circ f : N \to M_2$ are smooth, where p_1, p_2 are the projections onto the first and second factors, respectively. Show, moreover, that the product smooth structure is the only smooth structure with this property.

12: If M, N are smooth manifolds, show that $M \times N$ is diffeomorphic to $N \times M$.

(*) 13: [Lee, problem 2.6] For M a (smooth) manifold, let $C(M)$ denote the set of continuous functions from M to $\mathbb{R}$, thought of as an algebra (i.e., a ring and a vector space over $\mathbb{R}$) with scalar multiplication by $\mathbb{R}$, and pointwise addition and multiplication. Let $C^\infty(M)$ be the subalgebra of smooth functions. If $F : M \to N$ is continuous, let $F^* : C(N) \to C(M)$ be given by $F^*(f) = f \circ F$.

(a) Show that F^* is a linear map.

(b) Show that F is smooth $\Leftrightarrow F^*(C^\infty(N)) \subseteq C^\infty(M)$.

(c) Suppose F is a homeomorphism. Show that F is a diffeomorphism $\Leftrightarrow F^* : C^\infty(N) \to C^\infty(M)$ is an isomorphism.