Math 445 Homework 9

Due whenever you feel like it (before November 26...)

33. [NZM, p.333, Problem 7.3.3] Show that if the continued fraction $[a_0, \ldots, a_n]$ has convergents h_i/k_i, then $k_i/k_{i-1} = [a_i, a_{i-1}, \ldots, a_0]$ for every $i \geq 1$.

34. [NZM, p.336, Problem 7.5.3 (sort of)] If $\alpha < \beta < \gamma$ are irrational numbers, $\alpha = [a_0, a_1 \ldots]$, $\beta = [b_0, b_1, \ldots]$, $\gamma = [c_0, c_1, \ldots ...]$, and $a_i = c_i$ for $0 \leq i \leq n$, then $a_i = b_i = c_i$ for $0 \leq i \leq n$.

    [Hint: Induction! Use $\alpha = [a_0, \ldots, a_{i-1}, a_i + r_i]$, etc. and Problem #30 to compare $a_{i+1} = \lfloor \frac{1}{r_i} \rfloor$, etc. Note that if $x < y$ then $\lfloor x \rfloor \leq \lfloor y \rfloor$.]

35. [NZM, p. 333, # 7.3.6] Let p be prime and suppose $u^2 \equiv -1 \pmod p$ (so $p \equiv 1 \pmod 4$). Let $[a_0, \ldots, a_n]$ be the continued fraction expansion of $\frac{u}{p}$, and let i be the largest integer with $k_i \leq \sqrt{p}$. Show that $|\frac{h_i}{k_i} - \frac{u}{p}| < \frac{1}{k_i\sqrt{p}}$, and $|h_i p - k_i u| < \sqrt{p}$. Setting $x = k_i$ and $y = h_i p - u k_i$, show that $p | x^2 + y^2$ and $x^2 + y^2 < 2p$, so $x^2 + y^2 = p$.

36. [NZM, p. 336, # 7.4.7] Show that if $x = [a_0, a_1, \ldots]$ is the simple continued fraction expansion of (the irrational number) x, with convergents h_n/k_n, then
$$k_n|k_{n-1}x - h_{n-1}| + k_{n-1}|k_n x - h_n| = 1$$
for all n. [Hint: figure out how to (correctly!) remove the absolute value signs.]