Math 445 Homework 8 Solutions

36. Let h_n/k_n (as usual) denote the n^{th} convergent of the continued fraction expansion of the irrational number x. Show by example that it is possible for $b < k_{n+1}$ and $\left|x-\frac{a}{b}\right| < \left|x-\frac{h_n}{k_n}\right|$.

    Most any irrational number works. E.g., $\sqrt{5} = [2, 4, 4, \ldots] = 2.236067977\ldots$, has convergents $\frac{2}{1} = 2$ and $\frac{9}{4} = 2.25$, but $\frac{7}{3} = 2.3333\ldots$ has $|\sqrt{5} - \frac{7}{3}| < |\sqrt{5} - 2|$ and $3 < 4$.

37. Show that for any $c > 2$, there are only finitely many pairs of integers a, b with $|\sqrt{2} - \frac{a}{b}| < \frac{1}{b^c}$.

    For any $c > 2$, $c - 2 > 0$, and so there is an integer $b_0 > 0$ with $b_0^{c-2} > 2$. Then $b \geq b_0$ implies $b^{c-2} \geq b_0^{c-2} > 2$. Then if such a b would work, $|\sqrt{2} - \frac{a}{b}| < \frac{1}{b^c} = \frac{1}{b^{c-2}} \cdot \frac{1}{b^2} < \frac{1}{2b^2}$, we have, from class, that $\frac{a}{b} = \frac{h_n}{k_n}$ for some n. So only finitely many denomenators *other than convergents* will work. For each of these denomenators, the fractions a/b are all $1/b$ apart, so at most 2 can be within $1/b$ of $\sqrt{2}$, so at most 2 are within $1/b^c < 1/b$. So only finitely many a/b are <u>not</u> convergents.

    To finish, we also need to show that only finitely many can <u>be</u> convergents. But we know that for any convergent $r_n \frac{h_n}{k_n}$, r_{n+2} is closer to $\sqrt{2}$, and on the same side of $\sqrt{2}$, as r_n. So $|\sqrt{2} - r_n| > |r_{n+2} - r_n| = |\frac{(-1)^n a_n}{k_{n+2}k_n}| = \frac{2}{k_{n+2}k_n}$ for $n \geq 2$ (by a result from class). But $k_{n+2} = 2k_{n+1} + k_n = 2(2k_n + k_{n-1}) + k_n = 5k_n + 2k_{n-1} < 7k_n$, since $k_{n-1} < k_n$. So $|\sqrt{2} - r_n| > \frac{2}{k_{n+2}k_n} > \frac{2}{7k_n^2}$. But for any $c > 2$, $\frac{2}{7k_n^2} < \frac{1}{k_n^c}$ for only finitely many n; we need $k_n^{c-2} < 7/2$, which is true only for $k_n < (7/2)^{1/(c-2)}$. So only finitely many k_n will work, with, as before, at most 2 numerators for each. So only only finitely many rational numbers will meet the stated bound.

38. Let p be prime and suppose $u^2 \equiv -1 \pmod{p}$ (so $p \equiv 1 \pmod{4}$). Let $[a_0, \ldots, a_n]$ be the continued fraction expansion of $\frac{u}{p}$, and let i be the largest integer with $k_i \leq \sqrt{p}$. Show that $|\frac{h_i}{k_i} - \frac{u}{p}| < \frac{1}{k_i\sqrt{p}}$, and $|h_ip - k_iu| < \sqrt{p}$. Setting $x = k_i$ and $y = h_ip - uk_i$, show that $p | x^2 + y^2$ and $x^2 + y^2 < 2p$, so $x^2 + y^2 = p$.

    We know that $|\frac{h_i}{k_i} - \frac{u}{p}| < |\frac{h_i}{k_i} - \frac{h_{i+1}}{k_{i+1}}| = \frac{1}{k_ik_{i+1}} < \frac{1}{k_i\sqrt{p}}$, by the choice of i. So, $|h_ip - k_iu| = |\frac{h_i}{k_i} - \frac{u}{p}|(k_ip) < \frac{1}{k_i\sqrt{p}}(k_ip) = \sqrt{p}$. If we set $x = k_i \geq 1$ and $y = h_ip - uk_i$, then $x^2 + y^2 = k_i^2 + (h_ip - uk_i)^2 < (\sqrt{p})^2 + (\sqrt{p})^2 = p + p = 2p$. And since $u^2 = pN - 1$ for some

N, $x^2+y^2 = k_i^2+(h_ip-uk_i)^2 = k_i^2(1+u^2)+p(ph_i^2-2uh_ik_i) = p(k_i^2N+ph_i^2-2uh_ik_i) \equiv 0 \pmod{p}$. So $0 < x^2+y^2 < 2p$ and is a multiple of p, so $x^2+y^2 = p$, as desired.

39. Show that for n a positive integer that is not a perfect square (translation: the continued fraction expansion of $\sqrt{n}$ never terminates), that at every stage of the continued fraction expansion of $x = \sqrt{n}$

$$x = [a_0, a_1, \dots, a_{k-1}, a_k + x_k]$$

x_k is always of the form $x_k = \dfrac{\sqrt{n}-c}{b}$, where $c, b \in \mathbb{N}$ and $b|n-c^2$. Conclude that the continued fraction expansion of $\sqrt{n}$ will eventually repeat, with a period of length at most $n\lfloor\sqrt{n}\rfloor$.

$\sqrt{n} = [\lfloor\sqrt{n}\rfloor + (\sqrt{n}-\lfloor\sqrt{n}\rfloor)]$, so $x_0 = \sqrt{n}-\lfloor\sqrt{n}\rfloor = \dfrac{\sqrt{n}-\lfloor\sqrt{n}\rfloor}{1}$ and $1|(n-(\lfloor\sqrt{n}\rfloor)^2$, as desired.

Continuing by induction, if we assume that $x_k = \dfrac{\sqrt{n}-c_k}{b_k}$, where $c_k, b_k \in \mathbb{N}$ and $b_k|n-c_k^2$, then writing $n-c_k^2 = b_kd_k = (\sqrt{n}-c_k)(\sqrt{n}+c_k)$, we have $a_{k+1} = \lfloor\dfrac{b_k}{\sqrt{n}-c_k}\rfloor = \lfloor\dfrac{\sqrt{n}+c_k}{d_k}\rfloor = N$ for some N, and then $x_{k+1} = \dfrac{\sqrt{n}+c_k}{d_k} - N = \dfrac{\sqrt{n}-(Nd_k-c_k)}{d_k} = \dfrac{\sqrt{n}-c_{k+1}}{b_{k+1}}$, where $c_{k+1} = Nd_k - c_k$ and $b_{k+1} = d_k$. To finish, we need to show that $b_{k+1}|n-c_{k+1}^2$, but $n-c_{k+1}^2 = n-(Nd_k-c_k)^2 = (n-c_k^2)+d_k(2Nc_k-N^2d_k) = b_kd_k + d_k(2Nc_k-N^2d_k) = d_k(b_k+2Nc_k-N^2d_k) = d_kM = b_{k+1}M$, as desired.

So by induction, for every $k \geq 0$, $x_k = \dfrac{\sqrt{n}-c}{b}$, where $c, b \in \mathbb{N}$ and $b|n-c^2$.