Math 445 Number Theory

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Computing $[a_0,\dots,a_n]$ from $[a_0,\dots,a_{n-1}]$:

$[a_0,\dots,a_n]=\dfrac{h_n}{k_n}$, where the h_n,k_n are defined inductively by

$h_{-2}=0,h_{-1}=1,k_{-2}=1,k_{-1}=0$, and $h_i=h_{i-1}a_i+h_{i-2}$, $k_i=k_{i-1}a_i+k_{i-2}$

The idea: induction! Check true for $i=0$. Suppose it is true for <u>any</u> continued fraction $[b_0,\dots,b_{n-1}]$. Then $[a_0,\dots,a_n]=[a_0,\dots,a_{n-2},a_{n-1}+\dfrac{1}{a_n}]$ has length n, so $[a_0,\dots,a_n]=$ $[a_0,\dots,a_{n-2},a_{n-1}+\dfrac{1}{a_n}]=\dfrac{H_{n-1}}{K_{n-1}}=\dfrac{h_{n-2}(a_{n-1}+\frac{1}{a_n})+h_{n-3}}{k_{n-2}(a_{n-1}+\frac{1}{a_n})+k_{n-3}}=\dfrac{h_{n-2}(a_{n-1}a_n+1)+h_{n-3}a_n}{k_{n-2}(a_{n-1}a_n+1)+k_{n-3}a_n}=$ $\dfrac{(h_{n-2}a_{n-1}+h_{n-3})a_n+h_{n-2}}{((k_{n-2}a_{n-1}+k_{n-3})a_n+k_{n-2}}=\dfrac{h_{n-1}a_n+h_{n-2}}{k_{n-1}a_n+k_{n-2}}=\dfrac{h_n}{k_n}$, as desired.

The real point here is that since $[a_0,\dots,a_n]$ and $[a_0,\dots,a_{n-2},a_{n-1}+\frac{1}{a_n}]$ both agree in the inductive definitions of their h_i and k_i, through $i=n-2$, this really *is* the calculation of the h_n/k_n for $[a_0,\dots,a_n]$.

There are several important things we can learn from this calculation. First, since $k_{-1}=0,k_0=0\cdot a_0+1=1$, and $k_n=k_{n-1}a_n+k_{n-2}\ge k_{n-1}+k_{n-2}>k_{n-1}$ for $n\ge 2$, the k_n are a strinctly increasing sequence of integers, and in fact, $k_n\ge n$. Even more, since $k_n\ge k_{n-1}+k_{n-2}$, the terms grow faster than the Fibonacci sequence (which has $F_n=F_{n-1}+F_{n-2},F_0=1,F_1=1$, and grows approximately like $\left(\dfrac{1+\sqrt{5}}{2}\right)^2$.

Second, $(h_n,k_n)=1$ for all n . In fact, $h_nk_{n-1}-h_{n-1}k_n=(-1)^n$ and $h_nk_{n-2}-h_{n-2}k_n=(-1)^na_n$.

This follows by induction; check $n=0$, and then $h_nk_{n-1}-h_{n-1}k_n=(h_{n-1}a_n+h_{n-2})k_{n-1}-h_{n-1}(k_{n-1}a_n+k_{n-2})=h_{n-1}k_{n-1}a_n+h_{n-2}k_{n-1}-h_{n-1}k_{n-1}a_n-h_{n-1}k_{n-2}=h_{n-2}k_{n-1}-h_{n-1}k_{n-2}=(-1)(h_{n-1}k_{n-2}-h_{n-2}k_{n-1})=(-1)(-1)^{n-2}=(-1)^{n-1}$, by induction, and then $h_nk_{n-2}-h_{n-2}k_n=(h_{n-1}a_n+h_{n-2})k_{n-2}-h_{n-2}(k_{n-1}a_n+k_{n-2})=h_{n-1}k_{n-2}a_n+h_{n-2}k_{n-2}-h_{n-2}k_{n-1}a_n-h_{n-2}k_{n-2}=a_n(h_{n-1}k_{n-2}-h_{n-2}k_{n-1})=a_n(-1)^{n-2}=(-1)^na_n$. This in turn gives us:

Third: setting $r_n=[a_0,\dots,a_n]=\dfrac{h_n}{k_n}$, we have $r_n-r_{n-1}=\dfrac{h_n}{k_n}-\dfrac{h_{n-1}}{k_{n-1}}=\dfrac{h_nk_{n-1}-h_{n-1}k_n}{k_{n-1}k_n}=$ $\dfrac{(-1)^n}{k_{n-1}k_n}$ and similarly, $r_n-r_{n-2}=\dfrac{h_n}{k_n}-\dfrac{h_{n-2}}{k_{n-2}}=\dfrac{(-1)^na_n}{k_{n-2}k_n}$.

This tells us many things! Since the k_n's are all positive (and, in fact, increasing), if we look at the "even" terms, $r_0,r_2,r_4,\dots$, this is an increasing sequence. The odd terms, $r_1,r_3,r_5,\dots$ are a decreasing sequence. And since successive terms are getting closer to one another, we have that the sequence $\{r_n\}_{n=0}^\infty$ converges. We will denote its limit, of course, as $[a_0,a_1,\dots,a_n,\dots]$.

But converges to what? If the continued fraction came from our procedure for computing the expansion of a real number x :: $a_0=\lfloor x\rfloor$, $x_0=x-a_0$, and inductively $a_n=\lfloor 1/x_{n-1}\rfloor$, $x_n=(1/x_{n-1})-a_n$, we have $x=[a_0,\dots,a_{n-1},a_n+x_n]<[a_0,\dots,a_{n-1},a_n]$ for n odd, and $x>[a_0,\dots,a_{n-1},a_n]$ for n even (by induction!). So $r_{2n}<x<r_{2n+1}$, so r_n converges to x !

In particular, $|x-r_n|<|r_{n+1}-r_n|=|\dfrac{(-1)^n}{k_nk_{n+1}}|=\dfrac{1}{k_nk_{n+1}}\le\dfrac{1}{k_n^2a_{n+1}}$ so the r_n make good rational approximations to x.