Math 445 Number Theory

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Continued fractions: Another example: $\sqrt{77}$

$8 < \sqrt{77} < 9$. so:

$a_0 = \lfloor \sqrt{77} \rfloor = 8, r_0 = \sqrt{77} - 8$, $a_1 = \lfloor \frac{1}{\sqrt{77}-8} \rfloor = \lfloor \frac{\sqrt{77}+8}{13} \rfloor = 1$, $r_1 = \frac{\sqrt{77}+8}{13} - 1 = \frac{\sqrt{77}-5}{13}$, $a_2 = \lfloor \frac{13}{\sqrt{77}-5} \rfloor = \lfloor \frac{\sqrt{77}+5}{4} \rfloor = 3$, $r_2 = \frac{\sqrt{77}+5}{4} - 3 = \frac{\sqrt{77}-7}{4}$, $a_2 = \lfloor \frac{4}{\sqrt{77}-7} \rfloor = \lfloor \frac{\sqrt{77}+7}{7} \rfloor = 2$, $r_2 = \frac{\sqrt{77}+7}{7} - 2 = \frac{\sqrt{77}-7}{7}$, $a_3 = \lfloor \frac{7}{\sqrt{77}-7} \rfloor = \lfloor \frac{\sqrt{77}+7}{4} \rfloor = 3$, $r_3 = \frac{\sqrt{77}+7}{4} - 3 = \frac{\sqrt{77}-5}{4}$, $a_4 = \lfloor \frac{4}{\sqrt{77}-5} \rfloor = \lfloor \frac{\sqrt{77}+5}{13} \rfloor = 1$, $r_4 = \frac{\sqrt{77}+5}{13} - 1 = \frac{\sqrt{77}-8}{13}$, $a_5 = \lfloor \frac{13}{\sqrt{77}-8} \rfloor = \lfloor \frac{\sqrt{77}+8}{1} \rfloor = 16$, $r_5 = \frac{\sqrt{77}+8}{1} - 16 = \frac{\sqrt{77}-8}{1} = r_0$,
and then the process will repeat. So, $\sqrt{77} = [8,1,3,2,3,1,16,1,3,2,3,1,16,\dots] = [3,\overline{1,3,2,3,1,16}]$.

Some basic facts. Finite, simple, continued fraction $x = [a_0, a_1, \dots, a_n]$, $a_i \in \mathbb{N}$ for all $i \geq 1$; $a_0 \in \mathbb{Z}$.

A basic formula: $[a_0, a_1, \dots, a_n] = a_0 + \frac{1}{[a_1, \dots, a_n]}$.

x is a rational number. (Proof: induction on n.)

Because $a_n = (a_n - 1) + \frac{1}{1}$, $[a_0, a_1, \dots, a_n] = [a_0, a_1, \dots, a_n - 1, 1]$. But this is the only type of equality:

If $[a_0, a_1, \dots, a_n] = [b_0, b_1, \dots, b_m]$ with $a_n, b_m > 1$, then $n = m$ and $a_i = b_i$ for all i .

The idea:

$[a_0, a_1, \dots, a_n] = a_0 + \frac{1}{[a_1, \dots, a_n]}$, and $[a_1, \dots, a_n] > 1$, so $a_0 = \lfloor [a_0, a_1, \dots, a_n] \rfloor = \lfloor [b_0, b_1, \dots, b_m] \rfloor = b_0$. So $\frac{1}{[a_1, \dots, a_n]} = \frac{1}{[b_1, \dots, b_m]}$, so $[a_1, \dots, a_n] = [b_1, \dots, b_m]$.

Then continue by induction.

Our basic formulas will hold just as well if the a_i are not integers. Another basic formula that we will repeatedly use is

$$[a_0, \dots, a_{n-1}, a_n] = [a_0, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n}]$$

Computing $[a_0, \dots, a_n]$ from $[a_0, \dots, a_{n-1}]$:

$[a_0, \dots, a_n] = \frac{h_n}{k_n}$, where the h_n, k_n are defined inductively by

$h_{-2} = 0, h_{-1} = 1, k_{-2} = 1, k_{-1} = 0$, and $h_i = h_{i-1}a_i + h_{i-2}$, $k_i = k_{i-1}a_i + k_{i-2}$

Proof: next time.