Math 445 Number Theory

August 23 and 25, 2004

Number theory is about *finding* and *explaining* patterns in numbers.

Ulam Sprial:

36	35	34	33	32	31	30
37	16	15	14	13	12	29
38	17	4	3	2	11	28
39	18	5	0	1	10	27
40	19	6	7	8	9	26
41	20	21	22	23	24	25
42	43	44	45	46	47	48

Place the natural numbers in a rectangular spiral. The primes tend to fall on certain diagonal lines with more frequency than it seems they should?

This means: for certain values of α, β, γ, the sequences $n^2 + \alpha$, $n^2 + n + \beta$, $n^2 - n + \gamma$ have more primes than we *expect* them to.

Why? We don't yet know...

Modulus: $a \equiv b \pmod{n}$ means a and b leave the same remainder when you divide by n (i.e., n evenly divides $b - a$; we write $n | b - a$).

$317 = 11^2 + 14^2$, but 319 cannot be expressed that way. In fact,

$$\text{if } n = a^2 + b^2, \text{ then } n \equiv 0, 1, \text{ or } 2 \pmod{4}$$

We will explore *which ones* are a sum of two squares later on.

Similarly, if $n = a^3 + b^3 + c^3$, then $n \not\equiv 4, 5 \pmod{9}$. A conjecture (of "Waring type") states that

$$\text{if } n \not\equiv 4, 5 \pmod{9}, \text{ then } n = a^3 + b^3 + c^3$$

This is still unresolved.

Egyptian fractions:

Any rational number m/n can be written as a sum of reciprocals $1/a$ of integers. In fact, by repeatedly subtracting the largest reciprocal that we can from whatever is left, we find that

$$\frac{m}{n} = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$$

with $a_1 < a_2 < \ldots < a_k$ and $k \leq n$. But not every fraction $3/n$ can be expressed as a sum of *two* reciprocals (e.g, $3/7$). However, it is conjectured (the Erdös-Strauss Conjecture) that

$$\text{every fraction } \frac{4}{n} \text{ is the sum of at most 3 reciprocals.}$$

This has been verified to $n = 10^{14}$, but still remains open.