Math 417 Problem Set 7

Starred (*) problems are due Friday, April 1.

(*) 51. Recall (!) that a subgroup $H \leq G$ is <u>characteristic</u> if $\varphi(H) = H$ for every $\varphi \in \mathrm{Aut}(G)$. Show that if K is a characteristic subgroup of H and H is a characteristic subgroup of G, then K is a characteristic subgroup of G.

52. If G is a group, $g \in G$, and $g^{-1} \neq g$, and if $X = \{xgx^{-1} \ : \ x \in G\}$ contains an <u>odd</u> number of elements (in particular, it is finite), show that $g^{-1} \notin X$.

53. If G is an <u>abelian</u> group, let $K = \{a \in G \ : \ a^2 = 1\}$ and let $H = \{x^2 \ : x \in G\}$. Show that H and K are (normal) subgroups of G, and that $G/K \cong H$. [Hint: build a homomorphism $G \to H$ ...]

(*) 54. Show that if G_1 and G_2 are groups and $H_1 \leq G_1$ and $H_2 \leq G_2$ are normal subgroups, then $H_1 \oplus H_2 \leq G_1 \oplus G_2$ is a normal subgroup and $(G_1 \oplus G_2)/(H_1 \oplus H_2) \cong (G_1/H_1) \oplus (G_2/H_2)$.

55. (Gallian, p.202, # 31) Let $G = (\mathbb{R}^*, \cdot, 1)$ and $H = (\mathbb{R}^+, \cdot, 1)$ be the groups of non-zero and positive real numbers, respectively, under multiplication. Show that $G \cong H \oplus \mathbb{Z}_2$ by directly building an isomorphism (and its inverse). [Hint: the absolute value function will likely play a role...]

56. (Gallian, p.202, # 35 (sort of)) Show that the functions $\varphi, \psi : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Q}$ given by $\varphi(a, b, c) = 3^a 6^b 10^c$ and $\psi(a, b, c) = 3^a 6^b 12^c$ are homomorphisms (where the domains are groups under addition and the codomains are groups under multiplication), and that φ is injective, while ψ is <u>not</u>.

(*) 57. (Gallian, p.222, # 42) Show that if $N, K \leq G$ are <u>normal</u> subgroups of G and $K \leq N$, then N/K is a normal subgroup of G/K, and $(G/K)/(N/K) \cong G/N$. [This is the “Third Isomorphism Theorem” of Emmy Noether. One approach: start by looking at the ‘natural’ map $G \to G/N$.]

58. (Gallian, p.221, # 34) Show that there is a homomorphism $\varphi : \mathbb{Z}_{40}^* \to \mathbb{Z}_{40}^*$ so that $\varphi(11) = 11$ and $\ker(\varphi) = \{1, 9, 17, 33\}$.

59. If $H, K \leq G$ are <u>normal</u> subgroups of the group G, show that $H \cap K \subseteq G$ is also a normal subgroup of G, and there is an <u>injective</u> homomorphism $G/(H \cap K) \to (G/H) \oplus (G/K)$.