Math 417 Problem Set 1

Starred (*) problems are due Friday, January 29.

(*) 1. We have seen that rotation $R(\theta)$ by angle θ and reflection $S(\theta)$ in the line making angle θ are given in matrix terms as multiplication by

$$A(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \text{ and } B(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

Show that $S(\theta)\circ S(\psi)$ is a rotation, and $S(\theta)\circ R(\psi)$ and $R(\psi)\circ S(\theta)$ are both reflections, and determine which angle they rotate or reflect by.

2. (Gallian, p.38, #14) If we build a rhombus R (a quadrilateral with all four sides having equal length) by gluing two equilateral triangles together along a pair of sides, describe the symmetries of R in terms of rotations and reflections.

3. Describe the symmetries of a cylinder (of height h).

(*) 4. Show that the set $G = \{1, 5, 9, 13\}$ forms a group, with group multiplication being multiplication modulo 16. (One approach: build the 'Cayley' table! This helps to see why some needed properties hold.)

(*) 5. (Gallian, p.55, #18) Which elements $x \in D_4 =$ the group of symmetries of a regular 4-gon (i.e., square) satisfy $x^2 = e$? Which satisfy $x = y^2$ for some $y \in D_4$?

[Problem #1 can help you decide what an element y^2 can look like...]

6. (Gallian, p.56, #31) Show that for any group G, its 'Cayley' table is a *Latin square*: every group element appears exactly once in each row and column of the table.

7. (Gallian, p.57, #48) Show that the collection of all 3×3 matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$ forms a group under matrix multiplication.

[This group is known as the *Heisenberg group*, and arises in the study of one-dimensional quantum systems. You may find your row reduction prowess useful in finding inverses!]

8. Show that the collection of all $n \times n$ matrices A (with coefficients in your favorite ring R) which satisfy $A^T A = I$ (where A^T is the transpose of A and I is the $n \times n$ identity matrix) forms a group under matrix multiplication.

[This group is usually denoted $O_n(R)$ or $O(n)$ (if the ring R is "understood"), known as the *orthogonal group*.]