Math 417 Problem Set 9

Starred (*) problems are due Friday, November 9.

(*) 58. If $\varphi : G \to H$ is a surjective homomorphism and $N \leq G$ is a normal subgroup of G, show that $\varphi(N) \leq H$ is a normal subgroup of H. Show, on the other hand, that if φ is not surjective, then $\varphi(N)$ need not be a normal subgroup (hint: G is a normal subgroup of G !).

59. If $H, K \leq G$ are normal subgroups of the group G, show that $H \cap K \subseteq G$ is also a normal subgroup of G, and there is an injective homomorphism $G/(H \cap K) \to (G/H) \oplus (G/K)$.

[Note: This is a generalization of our work in class on $\mathbb{Z}_{21}$ and $\mathbb{Z}_{21}^*$.]

(*) 60. (Gallian, p.222, # 42) Show that if $N, K \leq G$ are normal subgroups of G and $K \leq N$, then N/K is a normal subgroup of G/K, and $(G/K)/(N/K) \cong G/N$. [This is the "Third Isomorphism Theorem" of Emmy Noether. One approach: start by looking at the 'natural' map $G \to G/N$.]

61. If G is a group, show that $H = \{(g, g) \ : \ g \in G\}$ is a normal subgroup of $G \oplus G \Leftrightarrow G$ is abelian; when H is normal, show that $(G \oplus G)/H$ is isomorphic to G.

[Hint: how would you build a homomorphism $G \oplus G \to G$ so that H would be the kernel? Note that at this point in the problem you can assume that G is abelian!]

62. (Gallian, p.202, # 35 (sort of)) Show that the functions $\varphi, \psi : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Q}$ given by $\varphi(a, b, c) = 3^a 6^b 10^c$ and $\psi(a, b, c) = 3^a 6^b 12^c$ are homomorphisms (where the domains are groups under addition and the codomains are groups under multiplication), and that φ is injective, while ψ is not.

(*) 63. (Gallian, p.202, # 37) If H is a normal subgroup in G and G is finite, and $g \in G$, show that the order of gH in G/H divides the order of g in G.

64. Show that in the symmetric group S_n, every commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ is an element of the subgroup A_n = the alternating group. Show, in addition, that every 3-cycle (a, b, c) can be written as a commutator $\alpha\beta\alpha^{-1}\beta^{-1}$. Conclude that every element of A_n can be written as a product of commutators.

65. Use problem #64 to show that if $\varphi : S_n \to G$ is a homomorphism from the symmetric group to an abelian group G, then $\varphi(A_n) = \{e_G\}$ and so $\varphi(S_n)$ is either the trivial subgroup or isomorphic to $\mathbb{Z}_2$.