Math 417 Problem Set 8

Starred (*) problems are due **either** Friday, October 19, if you want to get your graded worked back before the midterm, **or** Monday, October 22, if you don't care if you do or not.

(*) 51. (Gallian, p.209, #59) If $\varphi : G \to H$ is a homomorphism that is onto, show that if $g \in Z(G)$ then $\varphi(g) \in Z(H)$.

51. We showed that for $G = \mathbb{Z}[x]$ = the integer polynomials under addition, and $a \in \mathbb{Z}$, the function $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ given by $\varphi(p) = p(a)$ is a homomorphism. Describe the kernel of this homomorphism.

52. Suppose that H and K are (finite) subgroups of a group G and that $|H|$ and $|K|$ are relatively prime. Show that $H \cap K = \{e_G\}$.

53. Let H be a subgroup of G and let $a, b \in G$. Show that $aH = bH$ if and only if $Ha^{-1} = Hb^{-1}$.

(*) 54. (Gallian p.151, #10) Let a and b be elements of a group G, and let H and K be subgroups of G. If $aH = bK$, show that $H = K$.

[One approach: show, first, that $aH = bH$!]

(*) 55. (Gallian p.152, #33) Let H and K be subgroups of a finite group G with $K \subseteq H \subseteq G$. Prove that $[G : K] = [G : H] \cdot [H : K]$.

56. (Gallian p.153, #47) Show that in a finite group G with $|G|$ odd, for every $a \in G$ the equation $x^2 = a$ has exactly one solution.

[Hint: show that the function (not a homomorphism!) $f : G \to G$ given by $f(x) = x^2$ is onto !]

57. (Gallian p.151, #20) (Without directly computing $\varphi(n)$,) Show that if $n > 2$ then the group $\mathbb{Z}_n^*$ of units mod n (under multiplication) has even order.

[Hint: what is the order of $n - 1$?]