Math 417 Problem Set 6

Starred (*) problems are due Friday, October 5.

35. Show that the alternating group $A_8 \leq S_8$ contains an element of order 15. Does it have an element of order 14 ? Of order 16 ?

(*) 36. Show that every element of S_n can be written as a product of transpositions of the form $(1, k)$ for $2 \leq k \leq n$. (Assume that $n > 1$ so that you don't have to worry about the philosophical challenges of $S_1 = \{()\}$...)

[Hint: why is it enough to show that this is true for transpositions?]

37. (Gallian, p.115, #46) Show that in the symmetric group S_7, there is <u>no</u> element $x \in S_7$ so that $x^2 = (1, 2, 3, 4)$. On the other hand, find two distinct elements $y \in S_7$ so that $y^3 = (1, 2, 3, 4)$.

(*) 38. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$, thought of as a function frm the group $(\mathbb{R}, +, 0)$ of real numbers under addition to the group $(\mathbb{R}^+, *, 1)$ of positive real numbers under multiplication, is an isomorphism of groups.

39. Suppose that G is a *dihedral group* (i.e., a group of symmetries of some regular n-gon), and define the function $\varphi : G \to H = (\{-1, 1\}, *, 1)$ to the group H (isomorphic to $\mathbb{Z}_2$) by $\varphi(\text{rotation}) = 1$ and $\varphi(\text{reflection}) = -1$. Show that φ is a homomorphism.

[Hint: Problem Set # 1 !]

40. Show that if G_1, G_2 are groups, $H_1 \leq G_1$ is a subgroup of G_1, and $\varphi : G_1 \to G_2$ is a homomorphism, then $H_2 = \{\varphi(h) \ : \ h \in H_1\}$ (the *image* of H_1) is a subgroup of G_2.

41. Show that the function $\varphi : \mathbb{Z}_{16}^* \to \mathbb{Z}_{16}^*$ given by $\varphi(a) = a^3$ is an isomorphism. What about $a \mapsto a^5$? Or $a \mapsto a^7$?

(*) 42. (Gallian, p.133, # 32) Suppose that $\varphi : (\mathbb{Z}_{50}, +, 0) \to (\mathbb{Z}_{50}, +, 0)$ is an isomorphism and $\varphi(7) = 13$. Show that, for all x, $\varphi(x) = kx$ for a certain k, and find k !