# Math 417 Second Exam

**Guidelines:** The following problems constitute the second exam, which will take place Tuesday, Decmber 11, from 10:00-12:00noon, in Avery 108. You may (and probably must!) prepare solutions to the problems in advance of the exam. You may turn in these prepared solutions at any time prior to the exam time, in which case you need not attend the actual exam. In preparing your solutions, you may consult our text [Gallian's *Contemporary Abstract Algebra* (any edition)], your class notes, any papers/solutions handed out in (our...) class, and your (Math 417) instructor. No other sources may be consulted, and if you run into any information in the normal course of your daily life which appears to be relevant to the solution of any of the problems below, you should avoid following that information, to the greatest extent that your other obligations allow. Each problem letter (A,B,C,D,E) is worth equal credit.

A. A subgroup $H \leq G$ is called *maximal* if whenever $H \leq K \leq G$ and $K$ is a subgroup of $G$ then either $K = H$ or $K = G$.

A-1. Show that if $H \leq G$ and $[G : H] < \infty$ is prime, then $H$ is a maximal subgroup of $G$.

A-2. Give an example of a finite group $G$ and maximal subgroup $H \neq G$ so that $H$ is not a normal subgroup of $G$.

B. If $n$ is odd and $\alpha \in S_n$ is an $n$-cycle, $\alpha = (a_1, a_2, \ldots, a_n)$, show that no element of the centralizer $C(\alpha) = \{\beta \in S_n \ : \ \alpha\beta = \beta\alpha\}$ of $\alpha$ has order 2.

[Hint: What is $\alpha(a_i, a_j)\alpha^{-1}$ ? And what does an element of order 2 look like?]

Use this to show that the order of $C(\alpha)$ must be odd.

C. Let $G$ be a finite group, $N \triangleleft G$ a normal subgroup of $G$, and $H \leq G$ a subgroup of $G$. Suppose that $|H|$ and $|N|$ are relatively prime (ie., $\gcd(|H|, |N|) = 1$). Show that the quotient group $G/N$ contains a subgroup isomorphic to $H$.

[Hint: show that there is an injective homomomrphism $\varphi : H \to G/N$ .]

D. Show that if $G = G_1 \oplus G_2$ is the direct sum of two groups, then the center $Z(G)$ of $G$ is equal to $Z(G_1) \oplus Z(G_2)$.

E. Show that any group $G$ with $|G| = 595$ must have at least two normal Sylow subgroups.