Math 325 Problem Set 6

Due Monday, March 12

20. Show *directly* (i.e., without quoting "Cauchy implies convergent" and "convergent implies Cauchy") that if a_n and b_n are Cauchy sequences, then so are the sequences $c_n = a_n + b_n$ and $d_n = a_n b_n$. [Hint: for the second, you will need to use Cauchy implies bounded?]

21. [Lay, p.181, problem # 18.15] A sequence a_n is called *contractive* if for some constant $0 < k < 1$ we have $|s_{n+2} - s_{n+1}| < k|s_{n+1} - s_n|$ for all $n \in \mathbb{N}$. Show that every contractive sequence is Cauchy (and therefore converges).

    [Hint: By induction, $|s_{n+2} - s_{n+1}| < k^n|s_2 - s_1|$, and $\displaystyle\sum_{r=m+1}^{n} k^r$ is something we know the exact value of...]

22. [Lay, p.189, problem # 19.8, sort of] If b_n is a subsequence of the sequence a_n (so $b_n = a_{g(n)}$ for some strictly monotone increasing function $g : \mathbb{N} \to \mathbb{N}$) <u>and</u> a_n is a subsequence of the sequence b_n, show by example that we <u>need</u> <u>not</u> have $a_n = b_n$ for every n.

23. [Lay, p.189, problem # 19.13] If a_n and b_n are both bounded sequences, show that $a_n + b_n$ is also bounded. Then show that
$$\limsup_{n\to\infty}(a_n + b_n) \ \leq\ \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \ .$$
<u>Then</u> find examples of sequences for which this inequality is *strict* (i.e., equality does not hold). [Hint: For the middle part, one of your previous problems will help... For the last part, at least one of your sequences (both of them?) must <u>not</u> be convergent....]