Math 325 topics covered since Exam 2

Most times we do not compute integrals using Riemann sums! Instead we rely on *Fundamental Theorem of Calculus:* If $f : [a,b] \to \mathbb{R}$ is integrable, and F is an antiderivative of f, so $F'(x) = f(x)$ on the interval, then $\displaystyle\int_a^b f(x)\ dx = F(b) - F(a)$

Basic question: which functions <u>have</u> antiderivatives?

$f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$. Domain = those x for which the limit <u>exists</u>. Limit exist at a: we say f is *differentiable* at a. f is *differentiable on* D if it is differentiable at every point of D.

If f is differentiable at a then f is continuous at a.

Intermediate Value Theorem for Derivatives: If f is differentiable on $[a,b]$, then for every $c, d \in [a,b]$ if γ is between $f'(c)$ and $f'(d)$ then there is a w between c and d so that $f'(w) = \gamma$.

This result was established by first proving an analogous result about slopes of secants: if f is continuous on an interval I, then the set $S = \{\dfrac{f(x)-f(y)}{x-y} \ :\ x < Y \text{ and } x,y \in I\}$ is an interval in $\mathbb{R}$. The point is that f' <u>need</u> <u>not</u> <u>be</u> <u>continuous</u> for this to be true! In fact, though, this result tells us a lot about <u>how</u> f' can <u>fail</u> to be continuous:

If g is not cts at a, then either $\displaystyle\lim_{x\to a}$ exists but is not equal to $f(a)$ [hole], or $\displaystyle\lim_{x\to a^-}$ and $\displaystyle\lim_{x\to a^+}$ both exist but are not equal [jump], or one of these one-sided limits fails to exist [oscillation]. The result above implies that if f is differentiable on $[a,b]$ but f' fails to be cts at some c, then this failure <u>must</u> be by oscillation. So where f' fails to be cts, it <u>really</u> fails...

The other point is that a function (like one with a jump discontinuity) that fails the IVT4Derivs <u>cannot</u> have an antiderivative! So the Fund Thm cannot be applied to compute its integral...

From our study of limits, we know that if $f'(a)$ exists, then $f'(a) = \displaystyle\lim_{n\to\infty} \frac{f(a+\frac{1}{n})-f(a)}{1/n} = \lim_{n\to\infty} n(f(a+1/n)-f(a)) = \lim_{n\to\infty} f_n(a)$. So for each a, $f'(a)$ is the limit of the values $f_n(a)$ of the continuous functions f_n. Thus $f'(x)$ is a *pointwise limit* of a sequence of continuous functions.

Such a function is called *Baire class one.* These functions can, in fact, be very far from continuous; the function $f(x) = 0$ if $0 \le x < 1$ and $f(1) = 1$ is the pointwise limit of the functions $f_n(x) = x^n$, and is not cts at $x = 1$. More, our favorite function $g(x) = 0$ for x irrational and $= 1/q$ if $x = p/q$ is rational in lowest terms, is Baire class one. But a striking result (which we did not prove) states that a Baire class one function f has lots of pts of cty; for every $x \in D$ and $\epsilon > 0$ there is a $y \in D$ with $|x-y| < \epsilon$ and f is cts at y.

But the impression we took away from this is that "pointwise limits are useless; the limit does not behave much like the sequence of functions". But there is a different way to formulate convergence of functions, that fares much better.

A sequence of functions f_n (with domain D) converges <u>uniformly</u> to f if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that for every $x \in D$ we have $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$. [The point is that N is chosen 'uniformly' for every $x \in D$.] This is much better behaved:

If $f_n : D \to \mathbb{R}$ are all continuous and $f_n \to f$ uniformly, then f is cts.

If $f_n : D \to \mathbb{R}$ are all uniformly continuous and $f_n \to f$ uniformly, then f is uniformly continuous.

If $f_n : [a,b] \to \mathbb{R}$ are all integrable and $f_n \to f$ uniformly, then f is integrable, <u>and</u> $\displaystyle\int_a^b f(x)\ dx = \lim_{n\to\infty} \int_a^b f_n(x)\ dx$.

These results have, it turns out, far-reaching consequences, reaching into nearly every subject where calculus finds application. This is because they are what allow us to use *power series* in the ways that we do!

A power series is a function $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$, though of as the pointwise limit of the partial sums $P_N(x) = \displaystyle\sum_{n=0}^{N} a_n(x-x_0)^n$, which are polynomials. The starting point is

Weierstrass M-test: If $f_n : D \to \mathbb{R}$ is a sequence of functions, and for a sequence of constants M_n we have $|f_n(x)| \leq M_n$ for all x and $\displaystyle\sum_{n=0}^{\infty} M_n$ converges, then the sequence of functions $P_N(x) = \displaystyle\sum_{n=0}^{N} f_n(x)$ converges uniformly to $f(x) = \displaystyle\sum_{n=0}^{\infty} f_n(x)$.

Every power series $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a radius of convergence R; essentially, f converges on $(x_0 - R, x_0 + R)$, and diverges outside of $[x_0 - R, x_0 + R]$. The radius R is typically computed as $R = \lim \dfrac{a_n}{a_{n+1}}$. In particular, if $0 < R_0 < R$, then the series $\sum a_n R_0^n$ converges (it is the value of f at $x_0 + R_0$), and so the Weierstrass M-test tells us that on $[x_0 - R_0, x_0 + R_0]$ the convergence of the partial sums of the power series to f is <u>uniform</u>. So since $a_n(x - x_0)^n$ is continuous, uniformly continuous, and integrable on every closed interval $[a,b]$ contained in $[x_0 - R_0, x_0 + R_0]$, so is the power series f ! Even more, $\int_a^b f(x)\ dx$ can be computed as the limit of the integrals of the partial sums, which as a finite sum can be integrated term by term, the integral of f can be computed by integrating term by term!

But even more is true! It is not true that the derivative of a uniform limit of differentiable functions is the limit of their derivatives. In fact, the uniform limit of differentiable functions need not be differentiable (anywhere). This leads, in fact, to the construction of continuous functions that are nowhere differentiable. But:

Theorem: If $f_n : [a,b] \to \mathbb{R}$ are all differentiable (hence cts), and $f_n \to f$ uniformly on $[a,b]$ and $f'(x) \to g(x)$ uniformly on $[a,b]$, then f is differentiable on $[a,b]$ and $f'(x) = g(x)$

This result appeals to the Fund Thm of Calculus! With it, we can prove the other great result about power series. If $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n (x - x_0)^n$ has radius of convergence R then the power series $g(x) = \displaystyle\sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$ also has radius of convergence R. Since the partial sums of g are the term-by-term derivatives of partial sums of the power series of f, and both partial sums converge uniformly on $[x_0 - R_0, x_0 + R_0]$ for any $R_0 < R$, the above result implies that $f'(x) = g(x)$ on $[x_0 - R_0, x_0 + R_0]$ (and hence on $(x_0 - R, x_0 + R)$). That is, the derivative of a power series is obtained by term-by-term differentiation. This is the heart of, for example, power series solutions to differential equations, which leads to the power series for most 'special functions' (Bessel, Legendre, Hankel, hypergeometric, gamma, etc.) that appear throughout the sciences and engineering.