Math 325 Topics Sheet for Exam 2

[Technically, everything from the topics sheeet for exam 1, plus...]

$(a_n)_{n=1}^{\infty}$ is *Cauchy* if for every $\epsilon > 0$ there is an N so that $n, m \geq N$ implies $|a_n - a_m| < \epsilon$. Convergent sequences are Cauchy. And every Cauchy sequence is convergent! I.e., $(a_n)_{n=1}^{\infty}$ Cauchy implies that there is an L so that $a_n \to L$.

Finding L: Cauchy sequences are bounded. Then setting $b_n = \sup\{a_k : k \geq n\}$, the sequence b_n is monotone *decreasing*, and bounded below (by the lower bound of the entire sequence a_n), so b_n converges to some number L. Then $a_n \to L$, because for for any $\epsilon > 0$ there is an N so that for $n \geq N$ we have

$$|a_n - a_N| < \epsilon/3 \text{ and } |a_N - b_N| < \epsilon/3 \text{ and } |b_N - L| < \epsilon/3;$$

adding together gives $|a_n - L| < \epsilon$.

This technology leads to: for a bounded sequence $(a_n)_{n=1}^{\infty}$, if we set $A_n = \{a_k : k \geq n\}$, then the sequence $b_n = \sup(A_n)$ is monotone decreasing and the sequence $c_n = \inf(A_n)$ is monotone increasing. Since each is also bounded (since A is), they each converge. We denote their limits as $\text{limsup} a_n = \lim b_n$ and $\text{liminf} a_n = \lim c_n$. Since $c_n \leq b_n$ for every n, we know that $\text{liminf} a_n \leq \text{limsup} a_n$. In fact, we learned that

$(a_n)_{n=1}^{\infty}$ is convergent $\Leftrightarrow$ $\text{liminf} a_n = \text{limsup} a_n$.

Many of the results about convergent sequences also hold for liminf's and limsup's, suitably interpreted; for example $\text{limsup}(a_n + b_n) \leq \text{limsup} a_n + \text{limsup} b_n$. [Note: equality need not hold!] Also, if $a_n \to L$ and b_n is bounded, then $\text{limsup} a_n b_n = L \text{limsup} b_n$.

Subsequences: A subsequence amounts to choosing some of the terms of a sequence; formally, a subsequence is $a_{n_k} = a_{g(k)}$ for some strictly monotone increasing function $g : \mathbb{N} \to \mathbb{N}$. Subsequences inherit many of the same properties of the original sequence, for example, boundedness and convergence: if $a_n \to L$ then $a_{n_k} \to L$ for every subsequence of a_n. On the other hand,

Every bounded sequence has a monotonic subsequence. [But we cannot determine beforehand whether or not it will be increasing or descreasing.]

In particular, for any bounded sequence a_n there are subsequences converging to both $\text{limsup} a_n$ and to $\text{liminf} a_n$.

Functions of a real variable: Sequences are functions with domain $\mathbb{N}$. When we expand our allowed domains, $f : D \to \mathbb{R}$ for some $D \subseteq \mathbb{R}$, we can extend our notion of limit, as well.

$\lim_{x \to a} f(x) = L$ means that $|f(x) - L|$ is small, so long as $|x - a|$ is small enough. One feature: a need not be in the domain of f. If fact, even if is is, we do not care what value f takes there; our formal definition of the limit is

For every $\epsilon > 0$, there is a $\delta > 0$ so that $x \in D$ and $0 < |x - a| < \delta$ implies that $|f(x) - L| < \epsilon$.

One sticky point: such limits need not be unique! [If $x \in D$ and $0 < |x - a| < \delta$ is satified by no number, then L could be anything we want!] For that matter, the limit need not exist! If it does, we say that f converges at a. For this purpose, we generally restrict ourselves, in discussing limits, to *accumulation points* of D. c is an accumulation point of D if for every $\delta > 0$ there is an $x \in D$ with $0 < |x - c| < \delta$. [That is, no matter how clse to c we need to be, there are points of D other than c that are at least that close.] With this, if c is an accumulation point of D, then the limit of f at c (if it exists) is unique.

Leveraging our work on sequences, $\lim_{x \to a} f(x)$ can be computed using sequences. $\lim_{x \to a} f(x) = L$ if and only if for <u>every</u> sequence a_n with $a_n \to a$ <u>and</u> $a_n \neq a$ for every n, we have $f(a_n) \to L$.

Then most of our familiar results about limits of sequences carry over to functions: for example if $f(x) \to L$ and $g(x) \to M$ as $x \to a$, then $(f+g)(x) \to L+M$ and $(f \cdot g)(x) \to LM$.

Continuity: From calculus you are used to the idea that for many functions to compute its limit we "plug in". That is, $\lim_{x \to c} f(x) = f(c)$. We call such a function *continuous at c*. If it is not continuous at c, we say it is *discontinuous* at c. If $f : D \to \mathbb{R}$ is continuous at c for every $c \in D$, we say that it is continuous on D. Continuity can be described using ϵ's and δ's: f is continuous at c provided that for every $\epsilon > 0$ there is a $\delta > 0$ so that $|x - c| < \delta$ implies that $|f(x) - f(c)| < \epsilon$.

Because limits can be described using sequences, so can continuity. In particular we have that f is continuous at $c \in D$ if and only if for every sequence $a_n \to c$ with $a_n \in D$ for every n, we have $f(a_n) \to f(c)$.

This enables us to use results about sequences to prove results about continuous functions. For example, if f and g are both continuous at c then so are $f+g, f-g, f \cdot g$, and f/g (so long, for the last, as $g(c) \neq 0$).

Possibly the two most important results about continuous functions are:

Intermediate Value Theorem: If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and D lies between $f(a)$ and $f(b)$, then there is a $c \in [a,b]$ so that $f(c) = D$.

Extreme Value Theorem: If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then there are $c, d \in [a,b]$ so that $f(c) \leq f(x) \leq f(d)$ for every $x \in [a,b]$.

The IVT can be used in root-finding: if f is continuous on an interval and $f(\alpha) < 0 < f(\beta)$, then there is a root of f lying between α and β. By repeatedly narrowing the distance between α and β (like, for example, taking their midpoint), we can find succesively better approximations to the root.

The IVT also allows us to show that every (positive) real number has an n-th root, for any natural number n; $f(x) = x^n - c$ always has a root.

The EVT tells us that maxima and minima exist, for function defined on a closed interval. [Techniques of calculus tells us how to find them, for differentiable functions.]

Uniform Continuity: In many situations, continuity alone is not 'enough' to obtain the results that we might want. For example, for each $x \in [0,1]$ $f_n(x) = x^n \to 0$ if $x < 1$ and $\to 1$ if $x = 1$. Each of the functions involved is continuous, but their 'limit' is not! The 'problem' is that continuity is defined for each point: the $\delta > 0$ we find is chosen with knowledge of both $\epsilon > 0$ <u>and</u> the point $c \in D$ at which continuity is being studied. So δ is a function of both ϵ and c.

A <u>stronger</u> form of continuity is obtained by eliminating one of these dependences: $f : D \to \mathbb{R}$ is *uniformly continuous* on D if for every $\epsilon > 0$ there is a $\delta > 0$ so that $x, y \in D$ and $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. That is, δ depends ony on ϵ, not on which points are input to f. A uniformly continuous function is therefore continuous, but the opposite need not be true. $f(x) = 1/x$ is continuous on the interval $(0, \infty)$, but is not uniformly continuous on that interval.

But if the domain of f is a closed interval $[a, b]$, then continuity does imply uniform continuity. [Our proof relied on the fact that bounded sequences have convergent subsequences!]

Still other conditions imply uniform continuity. A function $f : D \to \mathbb{R}$ is *Lipschitz* if there is a constant M so that for every $x, y \in D$ we have $|f(x) - f(y)| \leq M \cdot |x - y|$. [Then $\delta = \epsilon/M$ will work for the corresponding ϵ.]

Integration: One use of uniform continuity arises in the construction of the integral of a function. If $f : [a, b] \to \mathbb{R}$ is bounded, then we can try to approximate the area under the graph of f using rectangles. If the rectangles all lie inside this region, then our intuition tells us that the sum of their areas will be less than the area of the region. If the rectangles completely cover the region, then the sum of their areas will be greater than the area of the region. These ideas underlie the formal notion of the *Riemann integral.*

Given a collection of points $a = x_0 < x_1 < \cdots < x_n = b$ (this is called a *partition* P of the interval $[a, b]$) we can define $m_i = \inf\{f(x) \ : \ x \in [x_i, x_{i+1}]\}$ and $M_i = \sup\{f(x) \ : \ x \in [x_i, x_{i+1}]\}$. Then our intuition suggests that $L(f, P) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i)$ and $U(f, P) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i)$ are, respectively, less than and greater than the area under f. These are *lower and upper Riemann sums* of f. Then

$L(f) = \sup\{L(f, P) \ : \text{P a partition of } [a, b]\}$ and $U(f) = \inf\{U(f, P) \ : \text{P a partition of } [a, b]\}$ satisfy

$$L(f) \leq \text{the area under } f \leq U(f) \ .$$

$L(f)$ is the largest approximation by rectangles that we can obtain from 'below', and $U(f)$ is the smallest approximation by rectangles that we can obtain from 'above'. [These are called the *lower Riemman integral* and the *upper Riemann integral* of f, respectively.] We say that f is *(Riemann) integrable* on $[a, b]$ provided $L(f) = U(f)$; this common value we denote $\int_a^b f(x)\,dx$.

Since for partitions $P \subseteq Q$ we have $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$, we have that f is integrable over $[a, b] \Leftrightarrow$ for every $\epsilon > 0$ there is a partition P so that $U(f, P) - L(f, P) < \epsilon$.

Not every function is Riemann integrable! For example, our favorite terrible function, $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \notin \mathbb{Q}$, is integrable over no interval (containing more than one point). But:

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$. [The proof uses uniform continuity of f in an essential way!]

A function f is monotone if it is either monotone increasing [$x \geq y$ implies that $f(x) \geq f(y)$] or is monotone decreasing [$x \geq y$ implies that $f(x) \leq f(y)$] for every pair of points in the domain. Then we showed that every function that is monotone on an interval $[a, b]$ is integrable over $[a, b]$.