Math 325 Problem Set 9

Starred (*) problems are due Friday, November 2.

52. (a) Use the product and chain rules to derive a general formula for the second derivative $(f \circ g)''(x)$ of a composition; you should assume that $f''(x)$ and $g''(x)$ both exist.

(b) Find a 'hybrid product-chain rule' to express the derivative $(f \circ (gh))'(x)$; you should assume that f, g and h are all differentiable.

(*) 53. (Belding and Mitchell, p.100, #11) Suppose that f is differentiable on $[a, b]$ with $f'(a) > 0$ and $f'(b) < 0$. Show that:

(a) Neither $f(a)$ nor $f(b)$ is a maximum value for f on $[a, b]$; that is, there is a $c \in (a, b)$ so that $f(a) < f(c)$ and $f(b) < f(c)$. [Hint: a previous problem will help...]

(b) Use this and Rolle's Theorem to show that there is a point $c \in (a, b)$ where $f'(c) = 0$.

54. (Belding and Mitchell, p.100, #12) Prove the **Intermediate Value Theorem for Derivatives**: If f is differentiable on $[a, b]$ and $f'(a) < k < f\prime(b)$, then there is a $c \in (a, b)$ with $f'(c) = k$.

[Hint: Consider the 'auxiliary' function $h(x) = kx - f(x)$ and apply the results of the preceding problem. Note that $f'(x)$ need not be continuous (although examples of this are tough to construct!), so we cannot 'just' apply IVT...!]

(*) 55. Use Rolle's Theorem to show, by induction, that a polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

of degree n has at most n distinct roots (i.e., solutions to $p(x) = 0$).

[Hint: If p has degree n, then p' has degree $n - 1$ ...]

56. Use the 'other' Inverse Function Theorem to show that for every $n \in \mathbb{N}$ the function $f(x) = x^{1/n}$ is differentiable on $D = (0, \infty)$, and find $f'(x)$. Then use this and the Chain Rule to find the derivative of $g(x) = x^{m/n}$ for every $m, n \in \mathbb{N}$.

(*) 57. Suppose that $f, g : [0, 1] \to \mathbb{R}$ are both continuous on $[0, 1]$, differentiable on $(0, 1)$, $f(0) = g(0)$, and $f'(x) > g'(x)$ for every $x \in (0, 1)$. Show that $f(x) > g(x)$ for all $x \in (0, 1]$.

58. Let $f(x) = x + 2x^2 \sin(1/x)$ for $x \neq 0$ and set $f(0) = 0$. Show that f is differentiable everywhere, and that $f'(0) = 1$, but there is no interval (a, b) contiaining 0 where f is an increasing function.