Math 325 Problem Set 4

Starred (*) problems are due Friday, September 21.

(*) 17. (Belding and Mitchell, p.36, #20)

(*) (a) Show that if $x, y, c \in \mathbb{R}$, $c > 0$, and $|x - y| < c$, then $|x| < |y| + c$.

(*) (b) Show that if $x, y \in \mathbb{R}$ and $|x - y| < \frac{|x|}{2}$, then $|y| > \frac{|x|}{2}$.

18. A set A is said to be *bounded away from* 0 if there is an $\epsilon > 0$ so that for every $x \in A$ we have $|x| > \epsilon$. Show that A is bounded away from 0 if and only if the set $B = \{\frac{1}{x} \mid x \in A\}$ is bounded.

[N.B. "P if and only if Q" means P implies Q <u>and</u> Q implies P; that is, there are two things to show!]

19. If we set $A = \{x \in \mathbb{R} \mid x^3 < 2\}$, show that A is bounded above, so has a supremum $\alpha = \sup(A)$. Then show (in a manner similar to our classroom demonstrations) that $\alpha^3 < 2$ is not possible. (If you are feeling like doing even more, show that $\alpha^3 > 2$ is also impossible! From that, we can conclude that $\alpha^3 = 2$.)

(*) 20. (Belding and Mitchell, p.22, #2) Show that if a set of real numbers S has a least upper bound α, then this least upper bound is <u>unique</u>. That is, if β is also a least upper bound for S, then $\alpha = \beta$. [Hint: what's the alternative?]

21. (Belding and Mitchell, p.23, #6) For subsets $A, B \subseteq \mathbb{R}$, we define their 'sum' as $A + B = \{a + b \ : \ a \in A, b \in B\}$.

Show that if A and B are both non-empty and bounded from above, then so is $A + B$, and

$$\text{lub}(A + B) = \text{lub}(A) + \text{lub}(B) \ .$$

[Hint: show that $\text{lub}(A) + \text{lub}(B)$ is an upper bound! Then worry about whether there might be a smaller one...]

(*) 22. (Belding and Mitchell, p.23, #4) Let $A = \{a_1, a_2, a_3, \ldots\} = \{a_n \ : \ n \in \mathbb{N}\}$ and $B = \{b_1, b_2, b_3, \ldots\} = \{b_n \ : \ n \in \mathbb{N}\}$ be two sequences of real numbers. Let $C = \{a_n + b_n \ : \ n \in \mathbb{N}\}$, the sequence of their sums.

(*) (a) Show that if A and B have least upper bounds α and β, respectively, then $\alpha + \beta$ is an upper bound for C.

(*) (b) Find an example showing that $\alpha + \beta$ <u>need</u> <u>not</u> <u>be</u> the least upper bound for C.

23. (Belding and Mitchell, p.23, #7) Show that if $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$, then there is an <u>irrational</u> number $c \notin \mathbb{Q}$ with $\alpha < c < \beta$.

[Hint: c could be a rational multiple of $\sqrt{2}$ (why is that not rational?). Or see the outline that the text provides!]