Math 325 Exam 1 practice problems

1. Show that if $x, y \geq 0$, then the *arithmetic mean* $m = \dfrac{x+y}{2}$ and the *geometric mean* $\mu = \sqrt{xy}$ of x and y always satisfies $m \geq \mu$.
   [Hint: show that $2(m-\mu)$ is a square!]

   Show by an example that this inequality can be strict (that is, $m > \mu$).

2. Show that $\alpha = \sqrt{2+\sqrt{7}}$ is **not** a rational number.

3. Use induction to show that for every $n \geq 1$,
$$a_n = \sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{n(3n+5)}{4(n+1)(n+2)} = f(n)$$
   (Hint: write out what $f(n+1)$ is; it will help.)

4. (a) Use induction to show that for all $n \geq 1$, $n(n+1)$ is divisible by 2.
(b): Use induction to show that for every n≥1, $n^3 + 5n$ is divisible by 6.

5. Use the rational roots theorrem to show that $\alpha = \sqrt{2} - \sqrt{5}$ is not a rational number.

6. Suppose that S and T are both non-empty subsets of the real line, and both are bounded from above. Show that if $lub(S) \leq lub(T)$, then $lub(S \cup T) = lub(T)$.

7. Show, directly from the ϵ-δ definition, that $f(x) = x^2 - 3x - 5$ is continuous at $x = a$ for every $a \in \mathbb{R}$.

8. Prove, directly from the definition of a limit, that
$$\lim_{x\to 1}(x^2 - 3x + 1) = -1$$

9. Find the following limits (you need not give ϵ-δ level explanations):

(a): $\displaystyle\lim_{x\to 1} \frac{x^3 - 3x^2 + x + 1}{x^2 - 3x + 2}$

(b): $\displaystyle\lim_{x\to -1} \frac{x^2 + 4x + 3}{x^3 - x^2 + x - 1}$

(c): $\displaystyle\lim_{x\to 2} \frac{(x+1)^2 - 9}{x^4 - 3x^2 - 3x + 2}$