Math 325 Topics Sheet for the third exam

(dont forget the topics sheets from the first two exams!)

Integration: One use of uniform continuity arises in the construction of the integral of a function. If $f : [a, b] \to \mathbb{R}$ is bounded, then we can try to approximate the area under the graph of f using rectangles. If the rectangles all lie inside this region, then our intuition tells us that the sum of their areas will be less than the area of the region. If the rectangles completely cover the region, then the sum of their areas will be greater than the area of the region. These ideas underlie the formal notion of the *Riemann integral.*

But our first approach allows us to pick the 'heights' of our rectangles arbitrarily, using function values. Here the idea is that if all of the rectangles are 'thin' enough, and the heights of the rectangles 'approximate' f, then the sum of their areas should 'approximate' the area under the graph of f. Formally, we say that a function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if there is a number L so that for any choice of *partition* $P = \{a = x_0 < x_1 < \cdots < x_n = b$, so long as the widths $x_i - x_{i-1}$ of the subintervals are all small, then any choice of $c_i \in [x_{i-1}, x_i]$ (a 'sampling', S) has $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ close to L. Setting $||P||$ =max$\{x_i - x_{i-1}\ :\ i = 1, \ldots, n\}$ = the norm of the partition P, wht we want is that for any $\epsilon > 0$ there is a $\delta > 0$ so that for any partition P with $||P|| < \delta$ and for any sampling S for P, we have $|R(f, P, S) - L| = |\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - L| < \epsilon$. We call $R(f, P, S)$ a *Riemman sum* for f on $[a, b]$ with partition P and sampling S.

If we can show this for some L, then we define $\int_a^b f(x)\, dx = L$. For example, we can show directly from this definition that $\int_0^2 x = 2$, by showing, for $f(x) = x$ and any partition P of $[0, 2]$, that with $S = \{\frac{x_{i-1}+x_i}{2}\} = \{c_i\}$, we have $R(f, P, S) = 2$, and for any other sampling $S' = \{d_i\}$ of P, if $||P|| < \delta$ then $|R(f, P, S') - 2| = |R(f, P, S') - R(f, P, S))| \leq \sum |f(c_i) - f(d_i))| \cdot |x_i - x_{i-1}| = \sum |c_i - d_i| cdot |x_i - x_{i-1}| < \sum \delta |x_i - x_{i-1}| = 2\delta <$ ϵ¡ if we choose $\delta = \epsilon/2$.

But, in practice, we often wish to establish the f is integrable without knowing what L is. This we can do, since if $R(f, P, S)$ and $R(f, Q, T)$ are both close to L, then they are close to one another. This leads to a 'Cauchy-like' condition:

f is integrable on $[a, b]$ if and only if, (*) for every $\epsilon > 0$, we can find a $\delta > 0$ so that for any two partitions P, Q with $||p||, ||q|| < \delta$, and for any pair of samplings, S for P and T for Q, we have $|R(f, P, S) - R(f, Q, T)| < \epsilon$. [The integral L can then be found by taking any collection of partitions P_n with $||P_n|| \to 0$ and any samplings S_n (e.g., left endpoint, or right endpoints, or midpoints! of the intervals) for P_n; $L = \lim_{n\to\infty} R(f, P_n, S_n)$.

But! dealing with two partitions at once poses certain technical challenges, which get in the way of our effectively using this condition. What we would like to do insteead is to show that integrability can be established by looking at one partition at a time; we would like to show that it is implied by:

(**) for any $\epsilon > 0$, we can find a $\delta > 0$ so that for any partition P with $||P|| < \delta$, and any pair of samplings S, T for P, we have $R(f, P, S) - R(f, P, T)| < \epsilon$.

This turns out to work! To see this, we need to introduce the upper Riemann sum $U(f, P)$ and lower Riemnn sum $L(f, P)$: given a partition P of the interval $[a, b]$) we can define $m_i = \inf\{f(x)\ :\ x \in [x_i, x_{i+1}]\}$ and $M_i = \sup\{f(x)\ :\ x \in [x_i, x_{i+1}]\}$. Then $m_i \leq f(c_i) \leq M_i$ for any choice of sampling, so $L(f, P) = \sum m_i(x_i - x_{i-1}) \leq \sum f(c_i)(x_i - x_{i-1}) = R(f, P, S) \leq \sum M)i(x_i - x_{i-1}) = U(f, P)$

Since we can always choose $\{c_i\} = S$ so that $M_i - f(c_i)$ is as small as we like (by choosing values close to the sup), and we can choose $\{d_i\} = T$ so that $f(d_i) - m_i$ is as small as we like, as can make the finite sum $\sum(f(c_i) - f(d_i)(x_i - x_{i-1}) = R(f,P,S) - R(f,P,T)$ as close to $\sum(M_)i - m_i)(x_i - x_{i-1}) = U(f,P) - L(f,P)$ as we like, so we can show that that (**) holds $\Leftrightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ so that for any partition P with $||P|| < \delta$ we have $U(f,P) - L(f,P) < \epsilon$.

But for partitions $P \subseteq Q$ we have $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$, since points of Q subdivide some of the suintervals for P, and sup's over smaller intervals go down (so $U(f,Q) \leq U(f,P)$) while inf's for subintervals go up (so $L(f,P) \leq L(f,Q)$).

With this in hand, we can show that (**) implies our 'Cauchy' condition (*): given two partitions P, Q and samplings S, T, $R(f,P,S) - R(f,Q,T) \leq U(f,P) - L(f,Q) = (U(f,P) - L(f,P)) + L(f,P) - U(f,Q) + (U(f,Q) - L(f,Q))$. But then since $U(f,P \cup Q) - L(f,P \cup Q) \geq 0$, we have $R(f,P,S) - R(f,Q,T) \leq (U(f,P) - L(f,P)) + (L(f,P) - L(f,P \cup Q)) + (U(f,P \cup Q) - U(f,Q)) + (U(f,Q) - L(f,Q))$. But! $L(f,P) - L(f,P \cup Q) \leq 0$ and $U(f,P \cup Q) - U(f,Q) \leq 0$, since $P, Q \subseteq P \cup Q$. So $R(f,P,S) - R(f,Q,T) \leq (U(f,P) - L(f,P)) + (U(f,Q) - L(f,Q))$ If we pick a $\delta > 0$ so that $||P|| < \delta$ implies that $U(f,P) - L(f,P) < \epsilon/2$, then we have:

$||P||, ||Q|| < \delta$ implies that $R(f,P,S) - R(f,Q,T) < \epsilon/2 + \epsilon/2 = \epsilon$. Reversing roles of P and Q (and running the same argument again) yields $R(f,Q,T) - R(f,P,S) < \epsilon$, so $||P||, ||Q|| < \delta$ implies that $|R(f,P,S) - R(f,Q,T)| < \epsilon$, so (**) implies the 'Cauchy' integrability criterion, so f is integrable!

Unfortunately, not every function is Riemann integrable! For example, our favorite terrible function, $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \notin \mathbb{Q}$, is integrable over no interval (containing more than one point). <u>But</u>:

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$. We can show, using that fact that f is also <u>uniformly</u> continuous, (using the δ from uniform continuity associated to $\epsilon/(b-a) > 0$ that $||P|| < \delta$ implies $M_i - m_i < \epsilon$ for all i, so $U(f,P) - L(f,P) < \sum[\epsilon/(b-a)] \cdot (x_i - x_{i-1}) = [\epsilon/(b-a)] \cdot (b-a) = \epsilon$

From these different ways of approaching integrability, we can establish some familiar integration results:

If f is integrable on $[a,b]$ and $a < c < b$, then f *is integrable on both* $[a,c]$ *and* $[c,b]$ *and* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx \int_a^b f(x)\,dx$. [The italicised part needed Riemann sums to establish rigorously; then we could choose any partition we wanted to compute the integrals.]

If f and g are integrable on $[a,b]$, then *both* $c \cdot f$ *and* $f + g$ *are integrable on* $[a,b]$, *and* $\int_a^b (cf)(x)\ dx = c\int_a^b f(x)\ dx$ and $\int_a^b (f+g)(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$. Also, the product $h(x) = f(x)g(x)$ is integrable on $[a,b]$ (although the integral of h cannot, in general, be determined from the intwegrals of f and g).

Most times we do not compute integrals using Riemann sums! Instead we rely on

The (First) Fundamental Theorem of Calculus: If $f : [a,b] \to \mathbb{R}$ is integrable, and F is an antiderivative of f, so $F'(x) = f(x)$ on the interval, then $\displaystyle\int_a^b f(x)\ dx = F(b) - F(a)$

Basic question: which functions <u>have</u> antiderivatives?

$f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}$. Domain = those x for which the limit <u>exists</u>. Limit exist at a: we say f is *differentiable* at a. f is *differentiable on* D if it is differentiable at every point of D.

If f is differentiable at a then f is continuous at a.

Intermediate Value Theorem for Derivatives: If f is differentiable on $[a, b]$, then for every $c, d \in [a, b]$ if γ is between $f'(c)$ and $f'(d)$ then there is a w between c and d so that $f'(w) = \gamma$.

This result was established by first proving an analogous result about slopes of secants: if f is continuous on an interval I, then the set $S = \{\dfrac{f(x) - f(y)}{x - y} \ : \ x < Y \text{ and } x, y \in I\}$ is an interval in $\mathbb{R}$. The point is that f' need not be continuous for this to be true! In fact, though, this result tells us a lot about how f' can fail to be continuous:

If g is not cts at a, then either $\lim\limits_{x\to a}$ exists but is not equal to $f(a)$ [hole], or $\lim\limits_{x\to a^-}$ and $\lim\limits_{x\to a^+}$ both exist but are not equal [jump], or one of these one-sided limits fails to exist [oscillation]. The result above implies that if f is differentiable on $[a, b]$ but f' fails to be cts at some c, then this failure must be by oscillation. So where f' fails to be cts, it really fails...

The other point is that a function (like one with a jump discontinuity) that fails the IVT4Derivs cannot have an antiderivative! So the Fund Thm cannot be applied to compute its integral...

But lots of our favorite functions do have antiderivatives:

The (Second) Fundamental Theorem of Calculus: If f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t)\ dt$, defined on $[a, b]$, has $F'(x) = f(x)$ on $[a, b]$.

Being able to 'manufacture' antiderivatives in this way allows us to create new functions!, which turn out to be very useful to us in a variety of ways.

For example, the 'right' way to introduce exponential and logarithmic functions is to start with a familiar function, like $f(x) = 1/x$, and then (using that we 'know' that $(\ln x)' = 1/x$ and $ln(1) = 0$) construct the logarithm as $\ln x = \displaystyle\int_1^x \frac{1}{t}\ dt$. This has domain $(0, \infty)$ (the larget interval on which $f(x) = 1/x$ is continuous) and has derivative $(\ln x)' = 1/x > 0$ on that interval, so $\ln x$ is increasing, so it has a (continuous and differentiable) inverse function, which we call $exp(x) = e^x$ (!). All of the familiar properties of logarithms can be deduced from this definition, using the properties of integrals (and derivatives).

Polynomial approximations: As an application of integration, using the integration by parts formula that we learn in calculus,

$$\int_a^b f(x)g'(x)\ dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\ dx$$

we can, by induuction, construct the n-th degree Taylor polynomials of a function $f(x)$ that has enough derivatives defined on an interval containing $x = a$:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

Consequently, if $|f^(n+1)(t)| \le M_n$ for all t bwteeeen a and x, then $|f(x) - \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k| \le M_n \frac{|x-a|^{n+1}}{(n+1)!}$. If particular, since $\dfrac{|x-a|^{n+1}}{(n+1)!}$ is small, when n is large enough, then so long as M_n doesn't get overly large, the Taylor polynomials will make good (polynomial) approximations for $f(x)$. For example, for the functions $\sin x$ and $\cos x$, $M_n = 1$ will work, and so the Taylor polynomials will make good approximations for these functions. And since for $f(x) = e^x$ the bounds $M_n = e^{|x|}$ will work, we again get good polynomial approximations.

Uniform convergence: This discussion can be generalized to introduce the notion of the convergence of a sequence $\{f_n\}$ of functions $f_n : D \to \mathbb{R}$ to a 'limit' function $f : D \to \mathbb{R}$. We say that $f_n \to f$ pointwise if for every $x \in D$ and any $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $n \geq N$ implies that $|f_n(x) - f(x)| < \epsilon$. In general, pointwise convergence is not a terribly useful concept, because many of our standard useful properties of functions are not inhereited by the limit function f, even if all of the functions f_n doo have the property. For example, anll of the functiions f_n could be continuous, but f is not, and all of the f_n could be integrable on $[a, b]$ but f is not. Even more, it might be that f is integrable, but the values $\int_a^b f_n(x)\ dx$ might not converge to the value $\int_a^b f(x)\ dx$.

The basic problem is that with pointwwise convergence $f_n(x)$ could get close to $f(x)$ at a much later point in the sequence than at some other point $f_n(y)$. This is waht allows the limit function to get 'torn apart'. Thi sis because when choosing an N, it is a function of both ϵ and x. The solution, like it was for continuity itself, is to make N independent of x. We say that f_n converges to f uniformly on D if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that for every $x \in D$ we have $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$. Then we have:

If $f_n : D \to \mathbb{R}$ are all continuous and $f_n \to f$ unformly on D, then $f : D \to \mathbb{R}$ is also continuous.
If $f_n : [a, b] \to \mathbb{R}$ are all integrable on $[a, b]$ and $f_n \to f$ unformly on $[a, b]$, then $f : [a, b] \to \mathbb{R}$ is also integrable and the sequence $\int_a^b f_n(x)\ dx$ converges to $\int_a^b f(x)\ dx$.
If $f_n : D \to \mathbb{R}$ are all differentiable and $f_n \to f$ unformly on D, and $f_n'(x) \to g(x)$ uniformly on D, then $f : D \to \mathbb{R}$ is also differentiable and $f'(x) = g(x)$.

This has consequences for Taylor/power series, since as we learn in calculus, every power series has a radius of convergence R, so that the power series converges (pointeise) for $|x - a| < R$. But if $0 < S < R$, then the series in fact converges uniformly on $|x - a| < S$, and so, for example, the derivatives of the Taylor polynomials of a function $f(x)$ converges (uniformly) to the derivative of f .