Math 325 Topics Sheet for Exam 2

[Technically, everything from the topics sheet for exam 1, plus...]

Continuity: From calculus you are used to the idea that for many functions to compute its limit we "plug in". That is, $\lim_{x \to c} f(x) = f(c)$. We call such a function *continuous at c*. If it is not continuous at c, we say it is *discontinuous* at c. If $f : D \to \mathbb{R}$ is continuous at c for every $c \in D$, we say that it is continuous on D. Continuity can be described using ϵ's and δ's: f is continuous at c provided that for every $\epsilon > 0$ there is a $\delta > 0$ so that $|x - c| < \delta$ implies that $|f(x) - f(c)| < \epsilon$.

Because continuity has to do with limits, and we understand how limits behave when we combine functions, we can deduce some general rules about how continuity behanves when we combine functions: the sum, difference, constant multiple, product and (when we avoid where the denominator is zero) quotient of continuous functions is itself continuous. In addition, if f is continuous at $x = a$, and g is continuous at $x = f(a)$, then $g \circ f$ is continuous at $x = a$.

Possibly the two most important results about continuous functions are:

Intermediate Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and D lies between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ so that $f(c) = D$.

Extreme Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then there are $c, d \in [a, b]$ so that $f(c) \leq f(x) \leq f(d)$ for every $x \in [a, b]$.

The IVT can be used in root-finding: if f is continuous on an interval and $f(\alpha) < 0 < f(\beta)$, then there is a root of f lying between α and β. By repeatedly narrowing the distance between α and β (like, for example, taking their midpoint), we can find succesively better approximations to the root.

The IVT also allows us to show that every (positive) real number has an n-th root, for any natural number n; $f(x) = x^n - c$ always has a root.

The EVT tells us that maxima and minima exist, for function defined on a closed interval. [Techniques of calculus tells us how to find them, for differentiable functions.] To prove EVT, we introduced a stronger form of continuity.

Uniform Continuity: In many situations, continuity alone is not 'enough' to obtain the results that we might want. For example, for each $x \in [0, 1]$ $f_n(x) = x^n \to 0$ if $x < 1$ and $\to 1$ if $x = 1$. Each of the functions involved is continuous, but their 'limit' is not! The 'problem' is that continuity is defined for each point: the $\delta > 0$ we find is chosen with knowledge of both $\epsilon > 0$ and the point $c \in D$ at which continuity is being studied. So δ is a function of both ϵ and c.

A stronger form of continuity is obtained by eliminating one of these dependences: $f : D \to \mathbb{R}$ is *uniformly continuous* on D if for every $\epsilon > 0$ there is a $\delta > 0$ so that $x, y \in D$ and $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. That is, δ depends ony on ϵ, not on which points are input to f. A uniformly continuous function is therefore continuous, but the opposite need not be true. $f(x) = 1/x$ is continuous on the interval $(0, \infty)$, but is not uniformly continuous on that interval.

But if the domain of f is a closed interval $[a, b]$, then continuity does imply uniform continuity. We established this using the **Heine-Borel Theorem:** For the interval $[a, b]$ if, for each $x \in [a, b]$ we choose an open interval U_x containing x, then there is a finite set of points $x_1, \ldots, x_n$ so that the union of $U_{x_1}, \ldots, U_{x_n}$ contains all of $[a, b]$ (that is, this finite collection of intervals *covers* $[a, b]$. This result in turn allowed us to show that every continuous function $f : [a, b] \to \mathbb{R}$ is bounded, that is, there are $M, N \in \mathbb{R}$ so that $M \leq f(x) \leq N$ for every $x \in [a, b]$. In fact, if $a, b \in \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then f is also bounded.

Still other conditions imply uniform continuity. A function $f : D \to \mathbb{R}$ is *Lipschitz* if there is a constant M so that for every $x, y \in D$ we have $|f(x) - f(y)| \le M \cdot |x - y|$. Such functions are all uniformly continuous. [$\delta = \epsilon/M$ will work for the corresponding ϵ.]

Inverse functions. Functions that are one-to-one have inverses. A continuous function $f : I \to \mathbb{R}$ that is one-to-one must be either monotonically increasing or monotonically decreasing (in the 'strong' sense: we cannot have $x < y$ and $f(x) = f(y)$). [This had a remarkably slick proof...]. But more importantly, as a result we have that a one-to-one continuous function has a continuous inverse. This is because if (say) f is increasing, then $g = f^{-1}$ is also increasing, and given $a \in f(I)$ and $\epsilon > 0$, we have $f(g(a)-\epsilon) < f(g(a)) = a < f(g(a)+\epsilon)$, so there is a $\delta > 0$ with $f(g(a)-\epsilon) < a-\delta < a+\delta < f(g(a)+\epsilon)$, so $|x-a| < \delta$ means $a-\delta < x < a+\delta$, so $f(g(a)-\epsilon) < f(g(x)) < f(g(a)+\epsilon)$, so (since g is increasing!) $g(a)-\epsilon = g(f(g(a)-\epsilon)) < g(f(g(x))) = g(x) < g(f(g(a)+\epsilon)) = g(a)+\epsilon$, so $g(a)-\epsilon < g(x) < g(a)+\epsilon$, that is, $|g(x)-g(a)| < \epsilon$.

This, in turn, tells us that many of our favorite functions are continuous. Since $f(x) = x^n$ is continuous, it is one-to-one (for $x \ge 0$ if n is even), its inverse $g(x) = x^{1/n}$ is continuous. Also, for example, $f(x) = x^5 + 5x^3 + 17x - 4$ is continuous and one-to-one (by calculus, or directly comparing output for $x < y$), so it's inverse, which we (probably) can't express in an 'elementary' way, is continuous!

Differentiation: Now that we have put limits of functions on a firm foundation, we can take a more precise look at *differential calculus.*

A fcn $f : I \to \mathbb{R}$ is *differentiable* at $a \in I$ if $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists; we call the limit $f'(a)$. This means:

Given any $\epsilon > 0$, we can find $\delta > 0$ so that $0 < |x-a| < \delta$ and $x \in I$ implies $|\dfrac{f(x)-f(a)}{x-a} - f'(a)| < \epsilon$.

If f is differentiable at evey $a \in I$ we say that f is differentiable on I. Using the familiar variation $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, we can treat $f'(a)$ as a function of a and write the derivative function $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, whose (implied) domain is everywhere that the limit exists.

The properties of limits then give us:

If f is differentiable at a, then f is continuous at a ($f(x) - f(a) = \dfrac{f(x)-f(a)}{x-a}(x-a)$).

$(f+g)'(x) = f'(x) + g'(x)$ (where both sides are defined)
$(cf)'(x) = cf'(x)$ (when one side is defined, then the other one is)
$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (where both sides are defined)
$(f/g)\prime(x) = (f'(x)g(x) - f(x)g'(x))/(g(x))^2$ (where both sides are defined)

These follow the same line of argument your calculus instructor would have followed.

But one differentiation rule that requires more care is the **Chain Rule**: if g is differentiable nt $x = a$ and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at $x = a$ and $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$. This is because the standard line of argument, writing

$$\frac{(f\circ g)(x) - (f\circ g)(a)}{x-a} = \frac{(f\circ g)(x) - (f\circ g)(a)}{g(x)-g(a)} \frac{g(x)-g(a)}{x-a},$$

assumes that $g(x) \neq g(a)$ for x near a. This is true if $g'(a) \neq 0$:

If $g'(a) \neq 0$, then $\dfrac{g(x)-g(a)}{x-a} \neq 0$ near a (and in fact has the same sign as $g'(a)$, near a).

But! if $g'(a) = 0$, we need to appeal to the $\epsilon - \delta$ definition directly, to show that $(f \circ g)'(a) = 0$.

The 'other' inverse function theorem: if f is one-to-one <u>and</u> differentiable on an interval I, then the inverse function $g(x) = f^{-1}(x)$ is also differentiable (on $f(I)$), and $g'(x) = 1/f'(g(x))$.
This follows directly from the chain rule (since $f(g(x)) = x$), <u>if</u> we know that g is differentiable! But this, again, requires a direct appeal to the $\epsilon - delta$ definition, by showing that

$$\frac{g(f(a)+h)-g(f(a))}{h} = \frac{k}{f(a+k)-f(a)}, \text{ where } k = k(h) = g(f(a)+h) - a \text{ (!)}$$

Then since $h \to 0$ implies that $k \to 0$ (<u>since</u> g is continuous!), we get our result.

The main result that makes differentiation so useful, and allows the derivative of f to tell us useful things about f, is the

Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b)¡ then there is a(t least one) $c \in (a, b)$ with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

The proof of MVT uses that fact that (1) at a max or min c of a differentiable function (except at an endpoint of the domain), $f'(c) = 0$, (2) if $f(a) = f(b)$ then one of the absolute max or absolute min c of f (which exist by EVT) lies in (a, b) (so $f'(c) = 0$: this is **Rolle's Theorem**), and (3) applying Rolle's Thoerem to the function

$$g(x) = f(x) - (f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

yields a $c \in (a, b)$ with $g'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$.

This is the result which tells us most of the important things we know about the derivative:

If $f'(x) > 0$ on an interval I, then f is increasing on I (although it is important to note that $f'(a) > 0$ does <u>not</u> imply that f is increasing near a...)
If $f'(x) < 0$ on an interval I, then f is decreasing on I
If $f'(x) = 0$ on an interval I, then f is constant on I
If $f'(x) = g'(x)$ on an interval I, then for some constant $c \in \mathbb{R}$ we have $g(x) = f(x) + c$ on I. [This leads us to the notion of an *anitderivative*, and the fact that two antiderivatives of the same function differ by a constant.]

A more sophisticated version of MVT will allow us to prove L'Hôpital's Rule. The **Cauchy Mean Value Theorem** asserts that if $f, : [a, b] \to \mathbb{R}$ are continuous functions on $[a, b]$, and are both differentiable on (a, b), then there is a $c \in (a, b)$ so that $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$. It follows from Rolle's Theorem, applied to the function $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$

With it, we can prove the 'real' **L'Hôpital's Rule:** If $f(x), g(x) \to 0$ as $x \to a$, f and g are differentiable near a, and $\dfrac{f'(x)}{g'(x)} \to L$ as $x \to a$, then $\dfrac{f(x)}{g(x)} \to L$ as $x \to a$.
This allows us to 'iteratively' apply L'Hôpital, to compute limits like

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sin x - (x - \frac{x^3}{6} + \frac{x^5}{120})}{x^6}$$

as $\displaystyle\lim_{x\to 0} \frac{f^{(6)}(x)}{g^{(6)}(x)}$ [working backwards to invoke L'Hôpital to show all of the intervening limits exist and are equal!].