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Name: Solutions

Math 314 Exam 2

Show all work. Include all steps necessary to arrive at an answer unaided by a mechanical computational device. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Find, using any method (other than psychic powers), the determinant of the matrix

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ 4 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ 4 & 3 & 4 \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} 1 & 3/2 & 5/2 \\ 3 & 1 & -1 \\ 4 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3/2 & 5/2 \\ 0 & -7/2 & -17/2 \\ 0 & -3 & -6 \end{pmatrix}$$

$$\xrightarrow{(-1)} \begin{pmatrix} 1 & 3/2 & 5/2 \\ 0 & -3\ \cancel{-7/2} & \cancel{-17/2}\ -6 \\ 0 & -7/2 & -17/2 \end{pmatrix} \xrightarrow{(-3)} \begin{pmatrix} 1 & 3/2 & 5/2 \\ 0 & 1 & 2 \\ 0 & -7/2 & -17/2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3/2 & 5/2 \\ 0 & 1 & 2 \\ 0 & 0 & -3/2 \end{pmatrix}$$

$(\det = (1)(1)(-3/2))$

So $\det(A) = (2)(-1)(-3)(-3/2) = \boxed{-9}$

or:

$$\det(A) = 2\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ 4 & 4 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix}$$

$$= 2(4-(-3)) - 3(12-(-4)) + 5(9-4)$$

$$= 2(7) - 3(16) + 5(5) = 14 - 48 + 25 = 39 - 48 = \boxed{-9}$$

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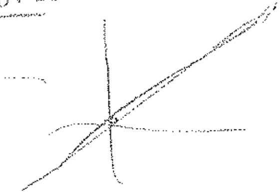
2. (20 pts.) Explain why the collection of vectors

x-y plane!



$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \textbf{ either } z = 0 \textbf{ or } x = y \right\}$$

(with addition and scalar multiplication taken from $\mathbb{R}^3$) is <u>not</u> a subspace of $\mathbb{R}^3$.

[Hint: what would this collection of vectors look like in $\mathbb{R}^3$?]

We can find vectors in W whose <u>sums</u> is <u>not</u> in W:

$$\cancel{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \quad \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{(x=y)} \in W, \quad \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{(z=0)} \in W \quad \text{but}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$ has <u>neither</u> $x=y$ <u>nor</u> $z=0$, so

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \notin W.$$

<u>or</u>: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are in W ($z=0$) <u>and</u> $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is, too ($x=y$)! So W contains all 3 standard coord vectors. If <u>W</u> were a subspace, it would contain all <u>lin</u> <u>combos</u> of those, i.e., <u>all</u> vectors in $\mathbb{R}^3$! So $W = \mathbb{R}^3$. But it <u>isn't</u>! Since $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ isn't in W (for example), W <u>can't</u> be $\mathbb{R}^3$, so it <u>can't</u> be a subspace...

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3. (25 pts.) Find bases for the row, column, and nullspaces of the matrix

$$A = \begin{pmatrix} 2 & 1 & 4 \\ -1 & 2 & 3 \\ 3 & -2 & -1 \\ 4 & 1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 4 \\ -1 & 2 & 3 \\ 3 & -2 & -1 \\ 4 & 1 & 6 \end{pmatrix} \to \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & -2 & -1 \\ 4 & 1 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -3 \\ 0 & 5 & 10 \\ 0 & 4 & 8 \\ 0 & 9 & 18 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 4 & 8 \\ 0 & 9 & 18 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Row space: transposes of non-$\vec{0}$ rows $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is a basis.

Column space: pivots in cols 1 & 2, use cols of $\underline{A}$. $\begin{pmatrix} 2 \\ -1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}$ is a basis

x_3 is free variable

Nullspace: $x_1 + x_3 = 0 \qquad x_1 = -x_3$

$x_2 = 2x_3 = 0 \qquad x_2 = -2x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ is a basis.

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4. (25 pts.) Find a collection of vectors from among the vectors

$$\begin{pmatrix}1\\-2\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}-2\\2\\1\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}$$

that forms a basis for $\mathbf{R}^3$, and express the remaining vectors as linear combinations of your chosen basis vectors.

[Hint: your work for the first part should tell you how to answer the second part!]

$$\begin{pmatrix}1&2&-2&3\\-2&1&2&1\\-1&2&1&2\end{pmatrix} \to \begin{pmatrix}1&2&-2&3\\0&5&-2&7\\0&4&-1&5\end{pmatrix} \to \begin{pmatrix}1&2&-2&3\\0&1&-2/5&7/5\\0&4&-1&5\end{pmatrix}$$

$$\to \begin{pmatrix}1&2&-2&3\\0&1&-2/5&7/5\\0&0&3/5&-3/5\end{pmatrix} \to \begin{pmatrix}1&2&-2&3\\0&1&-2/5&7/5\\0&0&1&-1\end{pmatrix} \to \begin{pmatrix}1&2&-2&3\\0&1&0&1\\0&0&1&-1\end{pmatrix}$$

$$\to \begin{pmatrix}1&2&0&1\\0&1&0&1\\0&0&1&-1\end{pmatrix} \to \begin{pmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&-1\end{pmatrix}$$

pivots in 1st 3 columns $\Rightarrow$ $\begin{pmatrix}1\\-2\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}-2\\2\\1\end{pmatrix}$ are a basis for $col(A) = \mathbb{R}^3$

b/c 3 = dim

the 4th column is a lin comb; find by reducing

$$\left(\begin{array}{ccc|c}1&2&-2&3\\-2&1&2&1\\-1&2&1&2\end{array}\right) \longrightarrow \begin{pmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&-1\end{pmatrix} ! \quad \text{So}$$

$$\boxed{v_4 = -\vec{v}_1 + \vec{v}_2 - \vec{v}_3} \; !$$

N.B. Any 3 of the vectors actually forms a basis for $\mathbb{R}^3$.

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5. (10 pts.) Explain why, if A is an $n \times n$ matrix and $A\vec{x} = \vec{0}$ has a non-trivial (i.e., $\vec{x} \neq \vec{0}$) solution, then it is <u>also</u> true that $A^T\vec{x} = \vec{0}$ has a non-trivial solution.

[Hint: think about invertibility, or row reduction, or rank, and what $A\vec{x} = \vec{0}$ says about this....]

$A\vec{y} = \vec{0}$ has a non-triv solution means that
row reducing A gives a free variable $\Rightarrow$
A has at most $n-1$ pivots. So A has a row of 0's,
cols.

so $\dim(\cancel{\text{col } A}\ \text{row}(A)) \leq n-1$
But $\text{row}(A) = \text{col}(A^T)$, so $\dim(\text{col}(A^T)) \leq n-1$, so
A^T, when row reduced, has $\leq n-1$ pivots.
so A^T has a free variable, and so $A^T\vec{x} = \vec{0}$ has a
non-trivial solution.

or: If it were the case that $A^T\vec{x} = \vec{0}$ didn't have a
non-trivial solution, then A^T has linearly independent
columns. But since A^T is a square matrix, this means
that A^T is invertible. But then $A = (A^T)^T$ is also
invertible. This means that A has linearly independent
columns, so $A\vec{x} = \vec{0}$ has no non-trivial solutions. But
it does! So $A^T\vec{x}$ must have non-trivial solutions.

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