Math 314/814 Matrix Theory
Exam 1 Practice Problems: Solutions

1. Use row reduction to find a solution to the following system of linear equations:

$$\begin{array}{rcrcrcr} -x & & & + & 2z & = & 3 \\ x & - & y & - & 3z & = & 3 \\ 2x & + & 3y & - & 3z & = & 6 \end{array}$$

We can express this as an augmented matrix and row reduce:

$$\begin{pmatrix} -1 & 0 & 2 & | & 3 \\ 1 & -1 & -3 & | & 3 \\ 2 & 3 & -3 & | & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & | & -3 \\ 1 & -1 & -3 & | & 3 \\ 2 & 3 & -3 & | & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & | & -3 \\ 0 & -1 & -1 & | & 6 \\ 0 & 3 & 1 & | & 12 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & -2 & | & -3 \\ 0 & 1 & 1 & | & -6 \\ 0 & 3 & 1 & | & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & | & -3 \\ 0 & 1 & 1 & | & -6 \\ 0 & 0 & -2 & | & 30 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & | & -3 \\ 0 & 1 & 1 & | & -6 \\ 0 & 0 & 1 & | & -15 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & | & -33 \\ 0 & 1 & 0 & | & 9 \\ 0 & 0 & 1 & | & -15 \end{pmatrix}$$

so our solution is $x = -33, y = 9, z = -15$.

2. Use row reduction to decide if the system of linear equations given by the augmented matrix:

$$(A|\mathbf{b}) = \left(\begin{array}{cccc|c} -1 & 0 & 2 & 3 & 2 \\ 3 & 2 & -4 & 1 & -2 \\ 0 & 3 & 0 & 1 & 6 \end{array}\right)$$

has a solution. If it does, does it have one or more than one solution?

We row reduce:

$$\left(\begin{array}{cccc|c} -1 & 0 & 2 & 3 & 2 \\ 3 & 2 & -4 & 1 & -2 \\ 0 & 3 & 0 & 1 & 6 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -2 & -3 & -2 \\ 0 & 1 & 1 & 5 & 2 \\ 0 & 3 & 0 & 1 & 6 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & -3 & -2 \\ 0 & 1 & 1 & 5 & 2 \\ 0 & 0 & -3 & -14 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -2 & -3 & -2 \\ 0 & 1 & 1 & 5 & 2 \\ 0 & 0 & 1 & 14/3 & 0 \end{array}\right)$$

This is enough to answer the question: the system of equations is consistent - there is no row of zeros in the coefficient matrix, in REF, opposite a non-zero number - so there is a solution. More, there is a free variable - the last column in the coefficient matrix - so there are more than one solution.

3. Is the vector $\vec{b} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ in the span of the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$?

More generally, what equation among a, b, c must hold in order for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to be in the span of $\vec{v}_1, \vec{v}_2$?

The first question asks if $\begin{pmatrix} 1 & 1 & | & -1 \\ 2 & 3 & | & 3 \\ 3 & -2 & | & 1 \end{pmatrix}$ has a solution. So we row reduce:

$$\begin{pmatrix} 1 & 1 & | & -1 \\ 2 & 3 & | & 3 \\ 3 & -2 & | & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & | & -1 \\ 0 & 1 & | & 5 \\ 0 & -5 & | & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & | & -6 \\ 0 & 1 & | & 5 \\ 0 & 0 & | & 29 \end{pmatrix}$$

This is inconsistent: the last row reads $0 = 29$. So there are no solutions, so $\vec{b}$ is not in the span of $\vec{v}_1$ and $\vec{v}_2$.

The second question asks <u>when</u> does $\begin{pmatrix} 1 & 1 & | & a \\ 2 & 3 & | & b \\ 3 & -2 & | & c \end{pmatrix}$ have a solution. We row reduce:

$$\begin{pmatrix} 1 & 1 & | & a \\ 2 & 3 & | & b \\ 3 & -2 & | & c \end{pmatrix} \to \begin{pmatrix} 1 & 1 & | & a \\ 0 & 1 & | & -2a+b \\ 0 & -5 & | & -3a+c \end{pmatrix} \to \begin{pmatrix} 1 & 0 & | & 3a-b \\ 0 & 1 & | & -2a+b \\ 0 & 0 & | & -13a+5b+c \end{pmatrix}$$

which will have a solution exactly when $-13a + 5b + c = 0$ (so that the system is consistent. So $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ will be in the span precisely when $-13a + 5b + c = 0$; this equation must hold in order to be in the span.

4. Use Gauss-Jordan elimination to find the inverse of the matrix A, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 5 & 8 \end{pmatrix}$$

We write down the superaugmented matrix and row reduce:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -5 & -2 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \\ 0 & -4 & -5 & -2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & -1 \\ 0 & -4 & -5 & -2 & 1 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -5 & 0 & 2 \\ 0 & 1 & 1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 10 & 1 & -4 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -5 & 0 & 2 \\ 0 & 1 & 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -10 & -1 & 4 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 1 & -2 \\ 0 & 1 & 0 & 13 & 1 & -5 \\ 0 & 0 & 1 & -10 & -1 & 4 \end{array}\right) \qquad \text{so } A^{-1} = \begin{pmatrix} 5 & 1 & -2 \\ 13 & 1 & -5 \\ -10 & -1 & 4 \end{pmatrix}$$

(b) Use your answer to find the solution to the equation $\quad Ax = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$A\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ has solution $\qquad \vec{x} = A^{-1}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & -2 \\ 13 & 1 & -5 \\ -10 & -1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 23 \\ -18 \end{pmatrix}$

5. Let $\mathbf{O}$ denote the $n \times n$ matrix with all entries equal to 0.

Suppose that A and B are $n \times n$ matrices with

$$AB = \mathbf{O} \ ,$$

but

$$B \neq \mathbf{O} \ .$$

Show that A **cannot** be invertible.
(Hint: suppose it *is*: what does that tell you about B ?)

If A has an inverse A^{-1}, then $A^{-1}A = I_n$, the $n \times n$ identity matrix. But then $B = I_nB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}\mathbf{O} = \mathbf{O}$, the last because multiplying a row by a column of 0s gives 0. So if A is invertible, then $B = \mathbf{O}$. But that's not true, so A can't be invertible.