Math 314/814, Section 6

Quiz number 2 Solutions

The system of linear equations

$$\begin{aligned} x + 2y + z - 3w &= 13 \\ 2x + y - 4z - w &= 10 \\ x \qquad -3z + 3w &= -3 \\ 2x + y - 4z - 8w &= 24 \end{aligned}$$

when written in matrix form, row reduces to

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 3 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(you need not show this!). Based on this, identify the pivot variable(s) and the free variable(s), and find three (3) different solutions to the system of equations.

When the matrix is in row echelon form, pivot variables correspond to the columns in which the first non-zero entry of each row occurs. These are the first, second, and fourth columns, and so the pivot variables are x, y, and w. The free variables are the "other" variables, i.e., they correspond to the columns that have no pivots in them. This is the third column, so the only free variable is z.

Because there is a free variable (and the system is consistent!), there are multiple solutions, which we can obtain by writing each pivot variable in terms of the free variable z, and specifying values for z. We have:

$x - 3z = 3$, $y + 2z = 2$, and $w = -2$ (but z free), so
$x = 3 + 3z$, $y = 2 - 2z$, z is free, and $w = -2$. So, for example:

$(z = 0)$: $x = 3$, $y = 2$, $z = 0$, $w = -2$ is a solution;
$(z = 1)$: $x = 6$, $y = 0$, $z = 1$, $w = -2$ is a soluition;
$(z = 4)$: $x = 15$, $y = -6$, $z = 4$, $w = -2$ is a solution.

N.B.: If we write the solutions in vector form,

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3+3z \\ 2-2z \\ z \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ -2 \end{pmatrix} + z \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

and view this from the perspective of multivariate calculus, this describes the solutions as a paramatrized line; point plus multiples of a direction vector.