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Name: Solutions

Math 314 Exam 1

Show all work. Include all steps necessary to arrive at an answer unaided by a mechanical computational device. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Use row reduction to find a solution to the following system of linear equations:

$$\begin{array}{rcrcrcr} 2x & - & y & + & 2z & = & 1 \\ x & + & y & - & 3z & = & 2 \\ 3x & - & y & + & z & = & 4 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -3 & 2 \\ 3 & -1 & 1 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & 1 \\ 3 & -1 & 1 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & +1 & -3 & 2 \\ 0 & -3 & 8 & -3 \\ 0 & -4 & 10 & -2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 0 & 1 & -8/3 & 1 \\ 0 & -4 & 10 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 0 & 1 & -8/3 & 1 \\ 0 & 0 & -2/3 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 1 \\ 0 & 1 & -8/3 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Check:

$$-7\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + (-3)\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7-6 \\ -7+9 \\ 7-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \checkmark$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix}}$$

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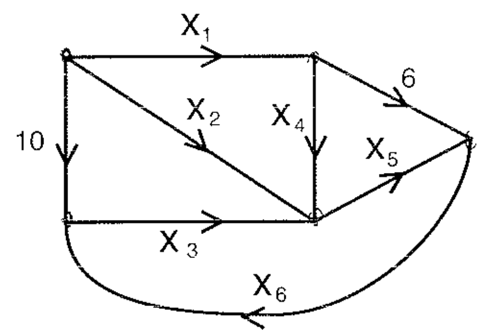
2. (20 pts.) The figure below shows a network of pipes with rates of flow through each pipe in the indicated directions marked. Assuming no leaks, construct the augmented matrix whose solutions give the values of the unknown flow rates. [To aid your instructor, please express the (underlying) equations using the variable order x_1 to x_6 implied by the notation...]



Flow in = flow out:

$x_1 + x_2 + 10 = 0$

$10 + x_6 = x_3$

$x_1 = x_4 + 6$

$x_2 + x_3 + x_4 = x_5$

$6 + x_5 = x_6$

$x_1 + x_2 = -10$

$x_3 - x_6 = 10$

$x_1 - x_4 = 6$

$x_2 + x_3 + x_4 - x_5 = 0$

$x_5 - x_6 = -6$

$$\left(\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & 0 & -1 & 10 \\ 1 & 0 & 0 & -1 & 0 & 0 & 6 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -6 \end{array}\right)$$

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3. (20 pts.) For which value(s) of x are the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ x \end{bmatrix}$$

linearly **dependent**?

Want: non-trivial solution.
$\rightarrow$ free variable.

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 2 & x & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & -8 & -5 & 0 \\ 0 & -10 & x-9 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & -10 & x-9 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & x-9+\frac{50}{8} & 0 \end{array}\right)$$

need: $x-9+\frac{50}{8}=0$

$\rightarrow x = 9-\frac{50}{8} = \frac{72-50}{8} = \frac{22}{8} = \frac{11}{4}$

For $x=\frac{11}{4}$, the vectors are linearly dependent.

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4. (25 pts.) Find the inverse of the matrix

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & -6 \\ 7 & 2 & -3 \end{pmatrix},$$

and use this inverse to find solutions to the systems of equations $A\vec{x} = \vec{b}$, for

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & -6 & 0 & 1 & 0 \\ 7 & 2 & -3 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 7 & 2 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & 0 & 1 & 0 \\ 0 & 1 & 20 & 1 & -3 & 0 \\ 0 & 2 & 39 & 0 & -7 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & 0 & 1 & 0 \\ 0 & 1 & 20 & 1 & -3 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & 0 & 1 & 0 \\ 0 & 1 & 20 & 1 & -3 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 12 & 7 & -6 \\ 0 & 1 & 0 & -39 & -23 & 20 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} 12 & 7 & -6 \\ -39 & -23 & 20 \\ 2 & 1 & -1 \end{pmatrix}}$$

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So, for those 3 $\vec{b}$'s, $A\vec{x} = \vec{b}$ has solutions

$$\begin{pmatrix} 12 & 7 & -6 \\ -39 & -23 & 20 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ -42 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 12 & 7 & -6 \\ -39 & -23 & 20 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -19 \\ 1 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 12 & 7 & -6 \\ -39 & -23 & 20 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -11 \\ 36 \\ -2 \end{pmatrix}$$

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5. (15 pts.) One of our characterizations of invertible matrices is that an $n \times n$ matrix A is invertible provided every system of equations $A\vec{x} = \vec{b}$ has a solution. Suppose that the cube of the matrix A, $A^3 = AAA$, is invertible. Explain why we can conclude that A must also be invertible. ~~Once we know that A is invertible, how can we express the inverse of A^3 in terms of the inverse of A~~?

A^3 invertible, so $A^3\vec{x} = \vec{b}$ has a solution

we <u>want</u> $A\vec{x} = \vec{b}$ to have a solution!

<u>But</u> $A^3\vec{x} = AAA\vec{x} = A(AA\vec{x}) = A(A^2\vec{x}) = \vec{b}$

<u>So</u> if we set $\vec{y} = A^2\vec{x}$, then $A\vec{y} = \vec{b}$, so

$A\vec{x} = \vec{b}$ <u>has</u> a solution. <u>So</u> A is invertible

[Removed from problem: once we know that A^{-1} makes sense, then $(A^3)^{-1} = (A^{-1})^3$, since $(A^3)(A^{-1})^3$

$$= AAAA^{-1}A^{-1}A^{-1} = AA(AA^{-1})A^{-1}A^{-1}$$
$$= AAA^{-1}A^{-1} = A(AA^{-1})A^{-1} = AA^{-1} = I_n \text{ !}$$
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