Math 221

Basic techniques for the second exam

Basic object of study: second order linear differential equations

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

Initial value problem:

$$y(t_0)=y_0 \qquad \text{and} \qquad y'(t_0)=y_0'$$

Basic fact: if $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval around t_0, then any initial value problem has a *unique* solution on that interval. Our Basic goal: find the solution!

Homogeneous: $g(t) = 0$ Constant coefficients: $p(t)$ and $q(t)$ are constant.

Start small: homogeneous with constant coefficients:

$$ay'' + by' + cy = 0$$

Basic idea: *guess* that $y = e^{rt}$, and plug in! Get:

$$(ar^2 + br + c)e^{rt} = 0 \quad \text{, so} \quad ar^2 + br + c = 0$$

Solve: get (typically) two roots r_1, r_2, so $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are *both* solutions.

The equation $ar^2 + br + c = 0$ is called the *characteristic equation* for our differential equation.

Operator notation: write $L[y] = y'' + p(t)y' + q(t)y$ (this is called a *linear operator*), then a solution to (*) is a function y with $L[y] = g(t)$.

For a linear differential equation, $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[yt]$, and so if y_1 and y_2 are both solutions to $L[y] = 0$ then so is $c_1y_1 + c_2y_2$. $c_1y_1 + c_2y_2$ is called a *linear combination* of y_1 and y_2. This is called the *Principle of Superposition*: more generally, if $L[y_1] = g_1(t)$ and $L[y_2] = g_2(t)$, then $L[y_1 + y_2] = g_1(t) + g_2(t)$.

With (the right) *two* solutions y_1, y_2 to a *homogeneous* equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (**)$$

we can solve *any* initial value problem, by choosing the right linear combination: we need to solve

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$
$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'$$

for the constants c_1 amd c_2; then $y = c_1y_1 + c_2y_2$ is our solution. This we can do directly, as a pair of linear equations, by solving one equation for one of the constants, and plugging into the other equation, or we can use the formulas

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. This makes it clear that a solution *exists* (i.e., we have the 'right' pair of functions), provided that the quantity

$$W = W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$$

W is called the *Wronskian* (determinant) of y_1 and y_2 at t_0 . The Wronskian is closely related to the concept of *linear independence* of a collection $y_1,\ldots,y_n$ of functions; such a

collection is linearly independent if the only linear combination $c_1y_1 + \cdots + c_ny_n$ which is equal to the 0 function is the one with $c_1 = \cdots = c_n = 0$.
Two functions y_1 and y_2 are linearly independent if their Wronksian is non-zero at *some* point; for a pair of solutions to (**), it turns out that the Wronskian is always equal to a constant multiple of

$$exp(-\int p(t)\ dt)$$

and so is either *always* 0 or *never* 0. We call a pair of linearly independent solutions to (**) a pair of *fundamental solutions*. By our above discussion, we can solve any initial value problem for (**) as a linear combination of fundamental solutions y_1 and y_2. By our existence and uniqueness result, this give us:

If y_1 and y_2 are a fundamental set of solutions to the differential equation (**), then *any* solution to (**) can be expressed as a linear combination $c_1y_1 + c_2y_2$ or y_1 and y_2.

So to solve an initial value problem for (**), all we need is a pair of fundamental solutions. For an equation with constant coefficients, we do this by finding the roots of the characteristic equation $ar^2 + br + c = 0$. We have the following basic facts:

If the roots of the characteristic equation are real and distinct, $r_1 \neq r_2$, then a fundamental set of solutions is

$$y_1 = e^{r_1t}, y_2 = e^{r_2t}$$

If the root of the characteristic equation are complex $\alpha \pm \beta i$, then a fundamental set of solutions is

$$y_1 = e^{\alpha t}\cos(\beta t), y_2 = e^{\alpha t}\sin(\beta t)$$

If the roots of the characteristic equation are *repeated* (and therefore real), $r_1 = r_2 = r$, then a fundamental set of solutions is

$$y_1 = e^{rt}, y_2 = te^{rt}$$

In showing the last of these facts, we introduced a general technique for finding a second, linearly independent, solution y_2 to (**), given a (non-zero) solution y_1; this was called *reduction of order*; if y_1 is a solution to (**), then so is

$$y_2 = y_1(t)\int \frac{\exp(-\int p(t)\ dt)}{(y_1(t))^2}\ dt$$

This formula arises by *assuming* that $y_2(t) = c(t)y_1(t)$, and then determining what differential equation $c(t)$ must satisfy! It turns out to be a first-order equation (hence the name reduction of order).

Much of what we just did for second order equation goes through without any change for even higher order (linear) equations:

$$L[y] = y^{(n)}\ = a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t) \qquad (!)$$

and its associated homogeneous equation

$$y^{(n)}\ = a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0 \qquad (!!)$$

In this case the correct notion of an initial value problem requires us to specify the values, at t_0, of y and all its derivatives up to the $(n-1)$st:

$$y(t_0) = y_0, y'(t_0) = y_0', \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

As with the second order case, we have a principle of superposition: $L[y_1] = g_1$ and $L[y_2] = g_2$, then $L[y_1 + y_2] = g_1 + g_2$. This means that linear combinations of solutions to the homogeneous equation (!!) are also solutions. And the general solution to (!!) can always be obtained (uniquely) as a linear combination of n *linearly independent* (or *fundamental*) solutions. Linear independence can be determined by computing a Wronskian determinant $W(y_1, \ldots, y_n)$.

The theory we developed for homogeneous equations with constant coefficients can be similarly extended. The equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

has a fundamental set of solutions determined by its characteristic equation

$$a_0 r^n + \cdots + a_{n-1} r + a_n = 0$$

Real roots r correspond to solutions $\exp(rt)$; complex roots to solutions $\exp(\alpha t)\cos(\beta t)$ and $\exp(\alpha t)\sin(\beta t)$. The only extra wrinkle is that we can have repeated roots which repeat *many* times, and even repeated complex roots! For each, we do as we did before and create new fundamental solutions by multiplying our basic solution by t, as many times as it repeats. For example, the equation

$$y^{(4)} + 2y'' + y = 0$$

has a characteristic equation with roots $i, i, -i$, and $-i$, and so its fundamental solutions are

$$\cos(t), t\cos(t), \sin(t), \text{ and } t\sin(t)$$

Our final concern is *inhomogeneous linear equations*

$$L[y] = y^{(n)} \ + a_1(t) y^{(n-1)} + \cdots + a_{n-1}(t) y' + a_n(t) y = g(t) \qquad (!)$$

with $g(t) \neq 0$. The principle of superposition tells us that for any pair of solutions Y_1, Y_2 to (!), $L[Y_2 - Y_1] = 0$, and so if we have a fundamental set of solutions to the associated homogeneous equation, $y_1, \ldots, y_n$, we can write

$$Y_2 = Y_1 + c_1 y_1 + \cdots + c_n y_n$$

In other words, we can find *any* solution to (!) by finding one *particular* solution, together with a fundamental set of solutions to the associated homogeneous equation (!!). Any initial value problem can then be solved by solving the system of equations

$$Y_1(t_0) + c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = g(t_0)$$
$$Y_1'(t_0) + c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) = g'(t_0)$$

all the way to

$$Y_1^{(n-1)}(t_0) + c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = g^{(n-1)}(t_0)$$

for the constants $c_1, \ldots, c_n$.

The only part of this we haven't really explored yet is finding a particular solution to (!). for this we have two techniques. The first is called the *Method of Undetermined Coefficients.*

Important: This works only for equations with constant coefficients!

The basic idea behind the technique is that for most kinds of functions, like polynomials, expoential, sines and cosines, or products of these, all of the functions derivatives are of the *same kind*. So if the function $g(t)$ in

$$L[y] = a_0 y^{(n)} \ + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \qquad (!)$$

is one of these kinds, what we do is *guess* that our solution y is the same kind. In particular,

If g is a polynomial of degree n, we set y to be a (different) polynomial of degree n,

If g is a multiple of an exponential $\exp(rt)$, we set y to be a a multiple $c\exp(rt)$ of g, If g is a multiple of $\sin(\beta t)$ or $\cos(\beta t)$, we set y to be a linear combination $a\sin(\beta t) + b\cos(\beta t)$,

If $g(t) = \exp(rt)\cos(\beta t)$ (or has a sine), we set y to be $a\exp(rt)\sin(\beta t) + b\exp(rt)\cos(\beta t)$,

If g is a polynomial of degree n times one of these, we set y to be a (different) polynomial of degree n time the corresponding function above.

Then we must plug this function into (!), and solve for the undetermined coefficients.

Of course, there is one wrinkle; sometimes our choice of y *cannot* work, because it is a solution to the associated *homogeneous* equation. For example, for the equation

$$L[y] = y'' + y = \cos(t)$$

The function $y = a\cos(t) + b\sin(t)$ will never solve it, because for such a function, $L[y] = 0$. In this case what we must do is multiply our guess by t, or more generally, by a high enough power to insure that our guess is not a solution to the homogeneous equation. For this, we must first determine the number of times the root which corresponds to our target solution occurs among the roots of the associated characteristic equation. This can be a trifle tricky to determine; for example, for the equation

$$y'' - 2y' + y = te^t$$

we should guess that our solution is $y = t(at + b)e^t$, since our original guess would be $y = (at + b)e^t$, but this is a solution to the homogeneous equation, while t times it is not; but for

$$y'' - 2y' + y = 3e^t$$

we should guess that our solution is $y = at^2e^t$, since te^t is still a solution to the homogeneous equation, but $y = t^2e^t$ is not.

Finally, if our function $g(t)$ is a linear combination of such functions, we can use this method to solve $L[y] =$ each piece, and then use the Principle of Superposition to find our solution by taking a linear combination.

Our other technique works (in theory) for any linear equation; we will restrict our attention to second order equations, for sake of simplicity. It is called variation of parameters, and starts with a pair of fundamental solutions y_1, yt to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (**)$$

and then *guess* that the solution to our inhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

is of the form $y(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$, and plug in. The resulting equation is too complicated, but if we make the simplifying assumption

$$c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0$$

then the equation becomes

$$c_1'(t)y_1'(t) + c_2'(t)y_2'(t) = g(t)$$

which we can solve:

$$c_1' = \frac{-gy_2}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \qquad c_2' = \frac{gy_1}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

Here again, the by now familiar Wronskian appears! Note that we must still *integrate* these functions, to determine c_1 and c_2 .