Math 221

Topics for first exam

Chapter 1: Introduction

§1: **The types of differential equations**

A differential equation is an equation involoving an (unknown) function y and some of its derivatives. The basic goal is to *solve* the equation, i.e., to determine which function or functions satisfy the equation. Differential equations come in several types, and our techniques for solving them will differ depending on the type.

Ordinary vs. partial: If y is a function of only one variable t, then our differential equation will involve only derivatives w.r.t. t, and we will call the equation an it ordinary differential equation. If y is a function of more than one variable, then our differential equation will involve partial derivatives, and we will call it a *partial differential equation.* We will deal almost exclusively with ordinary differential equations in this class.

Systems: Sometimes the rates of change of several functions are inter-related, as with the populations of a predator $y(t)$ and its prey $x(t)$, where $x' = ax - \alpha xy$ and $y' = \gamma xy - cy$. We call this a *system* of differential equations, and its solution would involve finding both $x(t)$ and $y(t)$.

Order: Techniques for solving differential equations differ depending upon how many derivatives of our unknown function are involved. The *order* of a differential equation is the order of the highest derivative appearing in the equation. The Implicit Function Theorem tells us that we can rewrite our equation so that it equates the highest order derivative with an expression involving lower order terms:

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}$$

Linear vs. non-linear: A differential equation is *linear* if it can be written as

$$a_0(t)y^{(n)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t)$$

(i.e., the function F is linear in the variables $y, y', \dots, y^{(n-1)}$, although it **need not** be linear in t). A differential equation is non-linear if it isn't linear! E.g.,

$$y' = y^2$$

is non-linear, while

$$y' = (\sin t)y/(1+t^2) - \cos(\cos t)$$

is linear.

In many cases, especially for first order differential equations, we can 'see' what a solution should look like without actually finding the solution. For first order equations, $y' = f(t, y)$, a solution $y(t)$ will satisfy $y'(t) = f(t, y(t))$, and so we can think of $f(t, y)$ as giving the *slope* of the tangent line to the graph of $y(t)$ at the point (t,y(t)). But since the function f is already known, we can draw small line segments at 'every' point of the t-y plane with slope $f(t, y)$ at the point (t, y); this is called the *direction field* for our differential equation. A solution to our differential equation is simply a function whose graph is tangent to each of these line segments at every point along the graph. Thinking of the direction field as a velocity vector field (always pointing to the right), our solution is then the path of a particle being push along by the velocity vector field. From this point of view it is not hard to believe that *every* (first order ordinary) differential equation has a solution, in fact many solutions; you just drop a particle in and watch where it goes. *Where* you drop it is important (it changes where it goes), which gives rise to the notion of an *initial value problem*; we seek to find the specific solution with the additional *initial value* $y(t_0) = y_0$.

Chapter 2: First order differential equations

§1: **Linear equations**

The most straightforward sort of differential equation to solve is the *first order linear ordinary differential equation*

$$y' = a(t)y + b(t)$$

We will typically (following tradition) write such equations as

$$y' + p(t)y = g(t)$$

For example, near the earth and in the presence of air resistance, the velocity v of a falling object obeys the differential equation $v' = g - kv$, where g and k are (positive) constants. There is a general technique for solving such equations, by trying to think of the left-hand side of the equation as the derivative of a single function. In form it look like the derivative of a product, and by introducing an *integrating factor* $\mu(t)$, we can actually arrange this. Writing

$$(\mu(t)y)' = \mu(t)(y' + p(t)y)) = \mu(t)g(t)$$

we find that (where exp(blah) means e raised to the power 'blah')

$$\mu(t) = \exp(\int p(t)\ dt)$$

and so

$$\mu(t)y = \int \mu(t)g(t)\ dt = \int(\exp(\int p(t)\ dt))\ g(t)\ dt + \mathrm{c}$$

which we can then solve for y.

For example, the differential equation $ty' - y = t^2 + 1$, after being rewritten in 'standard form as $y' - (1/t)y = t + (1/t)$, has integrating factor

$$\mu(t) = \exp(\int -1/t\ dt) = \exp(-\ln t) = 1/t$$

so we have

$$(1/t)y = \int 1 + 1/t^2\ dt = t - (1/t) + c$$

and so our solutions are $y = t^2 - 1 + ct$, where c is a constant.

But what is c ? Or solution is actually a *family* of solutions; a particular solution (i.e., a particular value for c) can be found from an initial value $y(t_0) = y_0$. For example, if we wished to solve the initial value problem

$$ty' - y = t^2 + 1\ ,\ y(2) = 5$$

we can plug $t = 2$ and $y = 5$ into our general solution to obtain $c = 1$.

§2: **Existence and uniqueness**

Our solution above can be expressed more concisely by saying that the solution(s) to

$$y' + p(t)y = g(t)$$

are

$$y = \exp(-\int p(t)\ dt)(\int \exp(-\int p(t)\ dt)g(t)\ dt + c)$$

where c is determined from an initial value. We can check that this actually *is* a solution by pluging it into our differential equation and seeing that it does in fact solve it.

We can in fact write down the solution to the added initial value problem $y(t_0) = y_0$ as

$$y = \exp(-\int_{t_0}^t p(x)\ dx)(\int_{t_0}^t \exp(\int_{t_0}^u p(v)\ dv)g(u)\ du + y_0)$$

although in practice this is more trouble than it is worth.

But this formula does tell us one important thing; for every first order linear ordinary differential equation, every initial value problem has a solution (we just wrote it down), and that solution is *unique* (because it has to *be* the one we wrote down). This is the basic existence and uniqueness theorem.

It actually says more (or less): the solution exists and is unique over any interval where the integrals we've written down make sense, which amount to any interval where *both* $p(t)$ and $g(t)$ are continuous. In other words, a solution to a linear differential equation can blow up or fail to be continuous only at the points of discontinuity of p or g .

§3: **Separable equations**

There is another class of first order equations for which we can readily find solutions by integration; there are the *separable* equations. A differential equation is separable if it can be written as

$$y' = A(t)B(y)$$

This allows us to 'separate the variables' and integrate with respect to dy and dt to get a solution:

$$\frac{1}{B(y)}dy = A(t)\ dt \text{ ; integrate both sides}$$

In the end, our solution looks like $F(y) = G(t) + c$, so it defines y *implicitly* as a function of t , rather than explicitly. In some cases we can invert F to get an explicit solution, but often we cannot.

For example, the separable equation $y' = ty^2$, $y(1) = 2$ has solution

$$\int \frac{dy}{y^2} = \int t\ dt + c$$

so solving the integrals we get $(-1/y) = (t^2/2) + c$, or $y = -2/(t^2 + 2c)$; setting $y = 2$ when $t = 1$ gives $c = -1$.

§4: **Linear versus non-linear equations**

There is a similar, but more restrictive, version of existence and uniqueness for non-linear first order differential equations:

If $y' = f(t, y)$ is a differential equation with both f and $\dfrac{\partial f}{\partial y}$ continuous for $a < t < b$ and $\alpha < y < \beta$, and $t_0 \in (a, b)$ and $y_0 \in (\alpha, \beta)$, then *for some* $h > 0$, the initial value problem

$$y' = f(t, y) \qquad , \qquad y(t_0) = y_0$$

has a unique solution for $t \in (t_0 - h, t_0 + h)$.

With non-linear equations, however, the size of the interval where we can guarantee existence (and uniqueness) can be very small, and often depends on the choice of initial value! For example, for the equation

$$y' = y^2$$

the righthand side is continuous everywhere (as is the partial derivative), but the interval we can choose for the solutions $y = -1/(t + c)$ depends on c, which will depend on the initial condition! And it can *never* be chosen to be the entire real line.

Failure to satisfy the hypotheses of the result can easily kill both existence and uniqueness. For example, the equation

$$y' = y^{1/3}$$

has many solutions with the initial condition $y(0) = 0$, such as $y = 0$ and $y = (2t/3)^{3/2}$.

§5: **Modeling with linear equations**

Radioactive decay: Our best model of the spontaneous decay of a radioactive substance assumes that every atom is as likely to decay at any given time as every other, so the rate of decay is proportional to the amount present. In other words, if $A(t) =$ the amount present at time t, then

$$A' = kA \qquad \text{for some (negative) constant } k$$

Soving this, we find that $A(t) = Ce^{kt}$ for some constant C. Using the initial value $A(0) = A_0$, we find that $A(t) = A_0 e^{kt}$.

Usually, we are not *given* k, but are given either the half-life (the time $t_{1/2}$ with $A(t_{1/2}) = (1/2)A(0)$), or, more generally, another data point $A(t_1) = A_1$. Plugging these into the equation above, we can solve for k .

Mixing problems: The basic setup has a solution of a known concentration mixing at a known rate with a solution in a vat, while the mixed solution is poured off at a known rate. The problem is to find the function which gives concentration in the vat at time t. It turns out that it is much easier to find a differential equation which describes the amount of solute (e.g., salt) in the solution (e.g., water), rather than the concentration.

If the concentration pouring in is A, at a rate of N, while the solution is pouring out at rate M with concentration $A(t)= x(t)/V(t)$, then if the initial volume is V_0, we can compute $V(t) = V_0 + (N-M)t$. The change in the amount $x(t)$ of solute can be computed as (rate falling in)$-$(rate falling out), which is

$$x' = AN - A(t)M = AN - \frac{x}{V_0 + (N-M)t}M$$

This is a linear equation, and so we can solve it using our techniques above. The integrating factor

$$exp(\int \frac{M}{V_0 + (N-M)t}dt)$$

is a power of the denomenator when $N - M \neq 0$, and is an exponential when $N = M$.

Newton's Law of Cooling: This states that the rate of change of the temperature $T(t)$ of an object is proportional to the difference between is temperature and the ambient temperature of the air around it. The constant of proportionality depends upon the particular object (and the medium, e.g., air or water) it is in. In other words,

$$T' = k(A - T)$$

Since a cold object will warm up, and a warm object will cool down, this means that the constant k should be positive. Writing the equation as

$$T' + kT = kA$$

which has solutions

$$T(t) = A + (T(0) - A)e^{-kt}$$

Typically, k is not given, but can be determined by knowing the temperature at some other time t_0, by plugging into the equation above and solving for k.

§6: **Population models and autonomous equations**

A differential equation is called *autonomous* if the function $f(t, y)$ is realy a function $f(y)$ only of the variable y. Such equations are separable, and so can be solved using the techniques outlined above. But we can learn alot about the solutions to such an equation simply by understanding the graph of $f(y)$.

One feature of the solutions is that we can translate in time and get another solution; if $y(t)$ is a solution to $y' = f(y)$, then so is $z(t) = y(t+c)$ for any constant c, as can be verified by plugging z into the differential equation. This can also be verified geometrically, using the direction field approach. For an autonomous equation, the slope of the direction field is always the same along horizontal lines (since it depends on y, not t), and so if we pick up a solution curve, tangent to the direction field, and translate it in the horizontal direction, it will *still* be everywhere tangent to the direction field, and so is also a solution. An example of such equations come from population models. The idea is that if y is the population at time t, then

$$y' = (\text{birth rate}) - (\text{death rate})$$

Typically, the birth rate is proportional to the population, i.e. is ry, while the death rate is proportional to the square of the population (this typically represents contact between individuals,arising from competition for food, overcrowding, etc.), i.e., is ky^2 . In other words,

$$y' = ry - ky^2$$

The key to understanding solutions to such equations $y' = f(y)$ is to find *equilibrium solutions*, that is, solutions $y =$ constant $=c$. Such solutions have derivative 0, and so for such solutions we must have $f(c) = 0$. The basic idea is that these equilibrium solutions tell us a great deal about the behavior of *every* solution to the differential equation.

If the function $f(y)$ is continuous, then between the zeroes of f (i.e., the equilibrium solutions of the differential equation) f has all the same sign, and so for the solutions, y' has the same sign, so $y(t)$ is either always increasing or always decreasing. It cannot *cross* the equilibrium solutions, since this would violate the uniqueness of solutions to our differential equation. (Here we assume that the derivative of f is also continuous.) If a solution curve becomes asymptotic to a horizontal line, that line must *be* an equilibrium solution, because the tangent lines along our solution must be becoming horizontal, i.e., $f(y(t)) = f(y)$ is approaching $0 = f(\text{limit of } y(t))$.

Therefore, the structure of the solutions is very simple; between consecutive equilibrium solutions, the solutions increase or decrease monotonically from one equilibrium to the other. This allows us to classify equilibrium solutions as one of three kinds: *stable equilibria*, where nearby solutions all converge *back* to the equilibrium, *unstable equilibria*, where nearby solutions all diverge away from the equilibrium, and *semistable equilibria*, where on one side the solutions converge back, and on the other they diverge away.

The easiest way to assemble this data is to plot the roots of f on a number line, and then determine the sign of f in the intervals in between. Where it is positive, solutions move to the right (i.e., up), while where it is negative they move left. Marking these as arrows, a stable equilibrium has arrows on both sides pointing towards it, and an unstable equilibrium has both arrows pointing away.

Chapter 8: Numerical methods differential equations

§1: The tangent line method

Most first order equations cannot be solved by the methods we have presented here. For such equations, the best we can often do is to *approximate* the solutions, using numerical techniques. The first of these techniques that were studied is the it tangent line method, also known as *Euler's method*. The idea is that our differential equation $y' = f(t, y)$ tells us the slope of the tangent line at every point of our solution, and the tangent line can be used to approximate the graph of a function, at least close to the point of tangency. In other words, for a solution to our differential equation,

$$y(t) \approx y(t_0) + y'(t_0)(t - t_0) = y_0 + f(t_0, y_0)(t - t_0)$$

for $t - t_0$ small. If we wish to approximate $y(t)$ for a value of t far away from our initial value t_0, we use the above idea in several steps. We cut up the interval into n pieces of length h (called the *stepsize*), and then set

$$\begin{array}{lcl} y_1 = y_0 + f(t_0, y_0)h & , & t_1 = t_0 + h \\ y_2 = y_1 + f(t_1, y_1)h & , & t_2 = t_1 + h \\ y_3 = y_2 + f(t_2, y_2)h & , & t_3 = t_2 + h \end{array}$$

and continue until we reach y_n, which will be our approximation to $y(t) = y(t_n)$. Each step can be thought of as a mid-course correction, using information about the direction field at each stage to determine which way the solution is tending.

Calculus teaches us that at each stage the error introduced is approximately proportional to the square of h. So with a stepsize half as large, we will require twice as many steps, but each introduces an error only about one-fourth as large, so overall we get an error only half as large. This leads us to conclude that as the stepsize goes to 0, the error between our approximate solution y_n and $y(t_n)$ goes to 0.

A slightly different technique is called *backwards Euler*, where we use the approximation

$$y(t_i) \approx y_{i+1} + y'(t_{i+1})(t_i - t_{i+1}) = y_{i+1} + f(t_{i+1}, y_{i+1})(-h)$$

and so we use the step

$$y_i = y_{i+1} - f(t_{i+1}, y_{i+1})h$$

This defines y_{i+1} implicitly in terms of y_i ; if we can solve this equation (essentially, by inverting the function on the right-hand side), we can then use this to determine a (different) y_n from the initial value y_0 . The analysis of how well this method does to approximate the true value of $y(t)$ is essentially the same as for (forward) Euler.