Math 221

Topics since the third exam

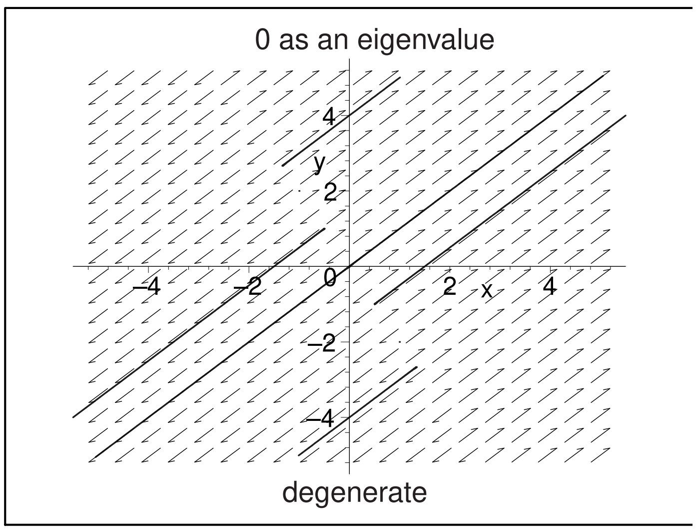
Chapter 9: Non-linear Systems of equations

§1: Typical Phase Portraits

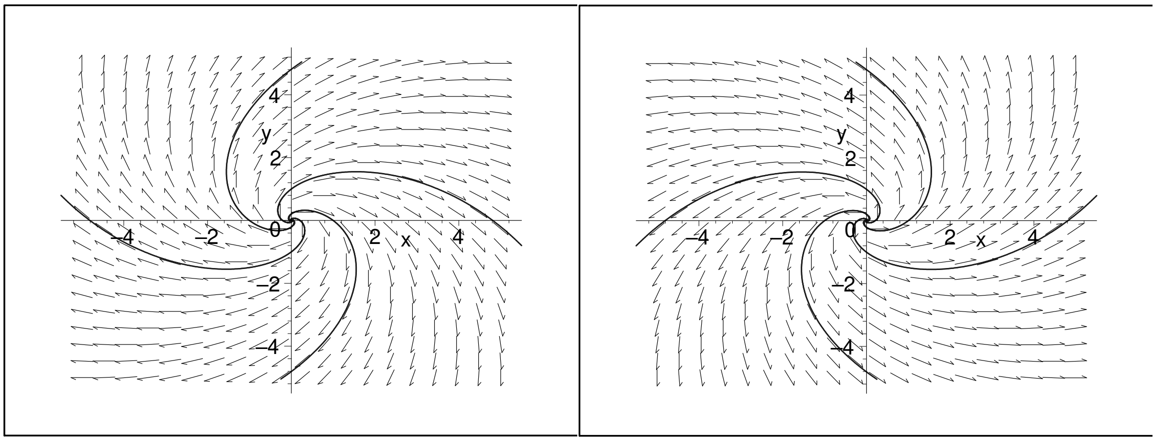
The structure of the solutions to a linear, constant coefficient, system of differential equations

$$\mathbf{u}' = A\mathbf{u}$$

is, essentially, completely determined by the eigenvalues of the associated matrix A. We call the origin (0,0) a *critical point* for the system, because it is the (typically, only) place where $\mathbf{u}'$=0 . We describe the critical point (0,0) by the behavior, or *phase portrait*, of the solutions around it. A table showing the nine typical pictures appears on the last page of these notes. The only portrait missing there is the case that one of the eigenvalues is zero. In this case, any multiple of the eigenvector with eigenvalue 0 is a critical point. We call this situation *degenerate*.



Everything about these portraits can be determined by knowing the eigenvalues and their eigenvectors, except, in the case of spirals and centers, the direction of the rotation. This can, however, be determined by evaluating $A\mathbf{u}$ at a point like (1,0), to determine if the trajectories are going up or going down at that point.



(§2,3,4,5: **Autonomous Systems**

Our determination of the above phase portraits can help us to understand many *non-linear* systems, particularly *autonomous systems.* These systems, just as in our one-variable situation, have right-hand sides that do not involve t, i.e.

$$x' = f(x, y) \qquad y' = g(x, y)$$

With such a system, we can talk about its *equilibrium solutions*, i.e., values of x and y where $x' = y' = 0$. These are determined by finding simultaneous solutions to the equations

$$f(x, y) = 0 \qquad g(x, y) = 0$$

Unlike with linear systems, there is usually more than one critical point for a non-linear system.

The basic idea behind our analysis of non-linear systems is that, near a critical point, the solutions to a non-linear system **look like** the solutions to some linear system, in a sense we will soon make precise. The idea is to *linearize* the system at the critical point, by replacing f and g by their linear approximations. At a critical point (x_0, y_0), the linear approximation to f is

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)[x - x_0] + f_y(x_0, y_0)[y - y_0] = f_x(x_0, y_0)[x - x_0] + f_y(x_0, y_0)[y - y_0]$$

(since we are at a critical point); g is similar. This allows us to approximate our system of equations by

$$x' = f_x(x_0, y_0)[x - x_0] + f_y(x_0, y_0)[y - y_0] \qquad y' = g_x(x_0, y_0)[x - x_0] + g_y(x_0, y_0)[y - y_0]$$

or, if we make the substitutions $u_1 = x - x_0$, $u_2 = y - y_0$, $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$, $c = g_x(x_0, y_0)$, $d = g_y(x_0, y_0)$, we get

$$u_1' = au_1 + bu_2 \qquad u_2' = cu_1 + du_2$$

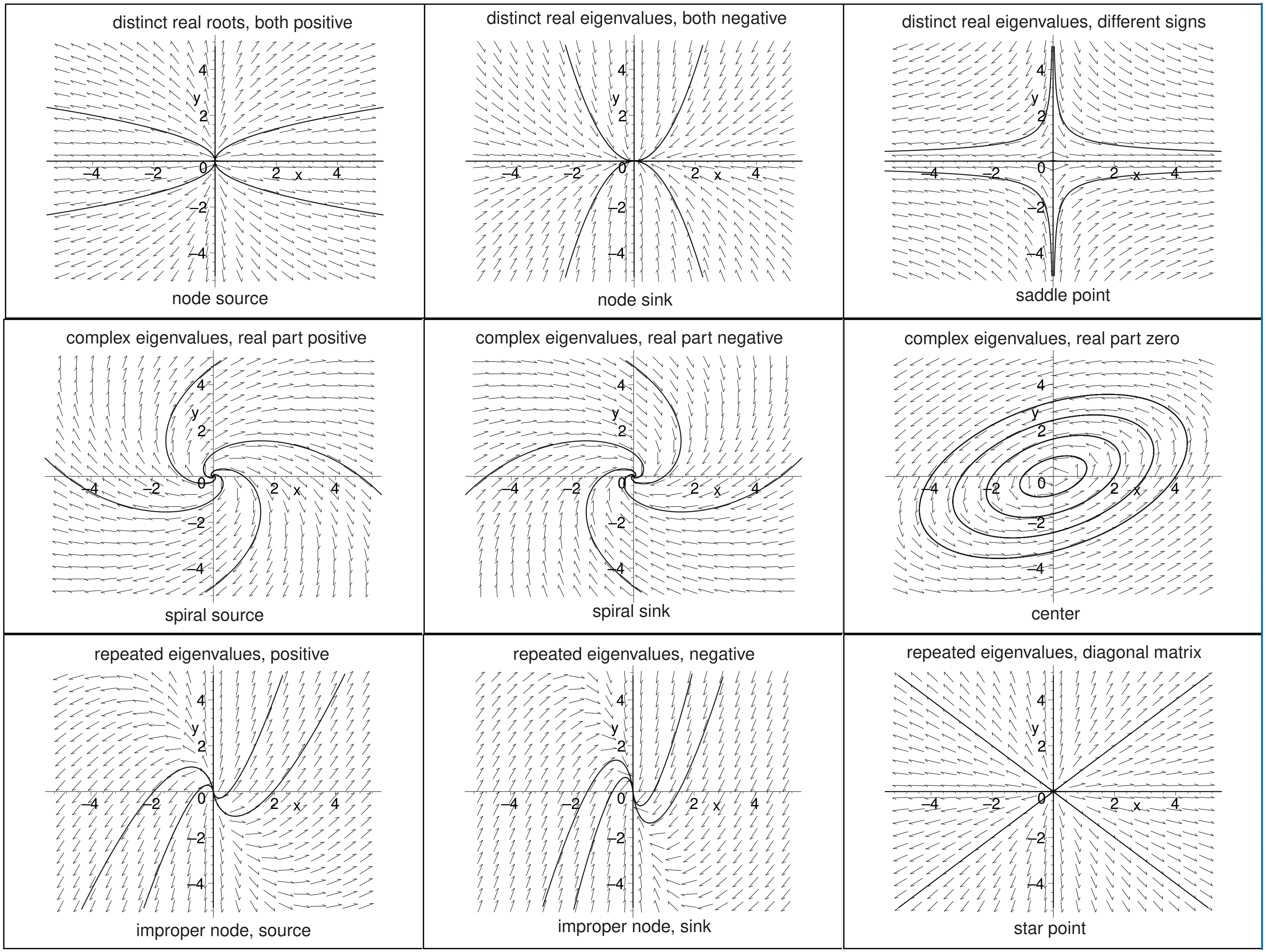
But this is precisely the kind of system of equations we know how to solve! The picture of the trajectories looks like one of our nine (or ten) phase portraits. The basic idea is that the phase portrait around a critical point of the non-linear system (usually) looks exactly like the portrait of its associated linearized system, at least in basic structure. In particular we can determined the *stability* of the critical point from these portraits, i.e., decide whether or not a solution that starts close to a critical point *stays* close to it (i.e., it is a sink, or source). This can be done for every critical point of the nonlinear system, one at a time.

In some situations, however, the phase portrait for the nonlinear system can be different from the linearized one. But this can only happen in the 'atypical' cases, where we have a center or an improper node. This is because our linearized system is only an *approximation*; and a center, for example, is a good approximation to a really, really tight spiral! The mathematical reason is that if you 'wiggle' the coefficients in a matrix a little bit, then if you started with distinct real eignevalues, or complex eigenvalues with non-zero real part, then your wiggled matrix will still have the same kind of solutions. But repeated real eigenvalues might change into distinct real ones, or even complex ones (although the stability of the critical point will remain the same, since the sign of the eigenvalues won't change. For the same reason, eigenvalues with zero real part might become ones with positive **or** negative real parts, so the nonlinear system might have a spiral source, spiral sink, **or** center critical point, instead of the center that the linearized system has.

Put more succinctly, if the linearized system has a node source, node sink, saddle, spiral source, or spiral sink, then so does the nonlinear system. The stability of the critical point therefore remains unchanged. But:

If the linear system has a center, the nonlinear one could have a spiral source, spiral sink, or center. Stability cannot be determined from the linearized system.

If the linear system has repeated eigenvalues (other than 0), then the nonlinear system could really exhibit any of our typical behaviors. However, the stability of the critical point will be the same as for the linearized system, which depends on whether the repeated eigenvalue is positive or negative.


distinct real roots, both positive
node source
distinct real eigenvalues, both negative
node sink
distinct real eigenvalues, different signs
saddle point
complex eigenvalues, real part positive
spiral source
complex eigenvalues, real part negative
spiral sink
complex eigenvalues, real part zero
center
repeated eigenvalues, positive
improper node, source
repeated eigenvalues, negative
improper node, sink
repeated eigenvalues, diagonal matrix
star point
x
y
–4
–2
0
2
4