Solutions

Name:

Math 221 Section 3

Final Exam

Show all work. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (10 pts.) Show that both of the functions

$$x(t) = 2t^{\frac{1}{2}} - 3 \qquad \text{and} \qquad x(t) = -3$$

are solutions to the initial value problem

$$2tx' - x = 3 \qquad x(0) = -3$$

Why doesn't this violate our existence/uniqueness theorem for IVPs?

$$x = 2t^{1/2} - 3$$
$$x' = 2 \cdot \tfrac{1}{2} t^{-1/2} = t^{-1/2}$$
$$2tx' - x = 2t\,t^{-1/2} - (2t^{1/2} - 3) = 2t^{1/2} - 2t^{1/2} + 3 = 3 \checkmark$$

$$x = -3$$
$$x' = 0$$
$$2tx' - x = 2t \cdot 0 - (-3) = 0 + 3 = 3 \checkmark$$

In standard form, our equation is

$$x' - \frac{1}{2t}x = \frac{3}{2t}$$

The functions $\frac{-1}{2t}$, $\frac{3}{2t}$ are not continuous at $t=0$. Our existence/uniqueness theorem only guarantees a unique solution on intervals where these functions are continuous. But there is no such interval around 0! So the result does not apply...

2. (10 pts.) Find the solution to the initial value problem

$$y' = \frac{3}{t}y + t^2 \qquad y(1) = 9$$

$$y' + \left(-\frac{3}{t}\right)y = t^2$$

$$y(t) = e^{-\int -\frac{3}{t}dt} \int t^2 e^{\int -\frac{3}{t}dt}\,dt$$

$$= e^{3\ln t} \int t^2 e^{-3\ln t}\,dt$$

$$= e^{\ln(t^3)} \int t^2 e^{\ln(t^{-3})}\,dt$$

$$= t^3 \left(\int t^2 t^{-3}\,dt\right) = t^3 \int t^{-1}\,dt$$

$$= t^3(\ln t + c) = t^3 \ln t + ct^3$$

$$9 = y(1) = 1^3 \ln 1 + c \cdot 1^3 = 0 + c = c$$

$$\boxed{y(t) = t^3 \ln t + 9t^3}$$

Check: $y' = t^2 + 3t^2 \ln t + 27t^2$

$\frac{3}{t}y + t^2 = 3t^2 \ln t + 27t^2 + t^2$ ✓

3. (a): (10 pts.) Find a set of fundamental solutions to the Cauchy-Euler equation

$$x^2y'' + 4xy' + 2y = 0\ .$$

$y = x^r$

$$r(r-1) + 4r + 2 = 0$$
$$r^2 + 3r + 2 = 0$$
$$(r+2)(r+1) = 0$$
$$r = -2, -1$$

$y_1 = x^{-2}$, $y_2 = x^{-1}$

$$W = \begin{vmatrix} x^{-2} & x^{-1} \\ -2x^{-3} & -x^{-2} \end{vmatrix}$$
$$= -x^{-4} + 2x^{-4} = x^{-4}$$

(b): (10 pts.) Use variation of parameters to find the solutions to the inhomogeneous equation

$$x^2y'' + 4xy' + 2y = e^x\ .$$

$$y'' + \frac{4}{x}y' + \frac{2}{x^2}y = x^{-2}e^x$$

$y = c_1y_1 + c_2y_2$

$u = x$ $dv = e^x dx$

$du = dx$ $v = e^x$

$$c_1 = \int \frac{-(x^{-1})(x^{-2}e^x)}{x^{-4}}\,dx = -\int xe^x\,dx$$
$$= -\left(xe^x - \int e^x\,dx\right)$$
$$= -(xe^x - e^x)$$

$$c_2 = \int \frac{(x^{-2})(x^{-2}e^x)}{x^{-4}}\,dx$$
$$= \int e^x\,dx = e^x$$

$$y = (-xe^x + e^x)x^{-2} + (e^x)x^{-1} = x^{-2}e^x$$

$$\boxed{y = x^{-2}e^x + c_1x^{-2} + c_2x^{-1}}$$

4. (15 pts.) Find the general solution to the differential equation

$$y''' + 2y' - 3y = \sin t$$

homogeneous solutions: $y''' + 2y' - 3y = 0$

$r^3 + 2r - 3 = 0$ $\quad 1 + 2 - 3 = 0$? ✓

$(r-1)(r^2 + r + 3) = 0$

$r = 1$ $\quad r = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 3}}{2} = \frac{-1}{2} \pm \frac{\sqrt{11}}{2} i$

$y_1 = e^t$, $y_2 = e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{11}}{2}t\right)$, $y_3 = e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{11}}{2}t\right)$

Undetermined coefficients:

$y = A\sin t + B\cos t$

$y' = A\cos t - B\sin t$

$y'' = -A\sin t - B\cos t$

$y''' = -A\cos t + B\sin t$

$\sin t = y''' + 2y' - 3y = (-A\cos t + B\sin t) + 2(A\cos t - B\sin t) - 3(A\sin t + B\cos t)$

$= (B - 2B - 3A)\sin t + (-A + 2A - 3B)\cos t$

$= (-B - 3A)\sin t + (A - 3B)\cos t$

$-B - 3A = 1$, $A - 3B = 0 \longrightarrow A = 3B$

$-B - 3(3B) = 1 = -10B$ $\quad B = -1/10$, $A = -3/10$

$$\boxed{y = c_1 e^t + c_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{11}}{2}t\right) + c_3 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{11}}{2}t\right) - \frac{3}{10}\sin t - \frac{1}{10}\cos t}$$

5. (15 pts.) Find the general solution to the system of equations

$$x' = -4x - 9y - t$$
$$y' = x + 2y$$

$$x = y' - 2y$$
$$x' = y'' - 2y'$$

$$y'' - 2y' = -4(y' - 2y) - 9y - t$$
$$y'' - 2y' = -4y' + 8y - 9y - t$$
$$y'' + 2y' + y = -t$$

homogeneous solution:

$$y'' + 2y' + y = 0$$
$$r^2 + 2r + 1 = 0$$
$$(r+1)^2 = 0$$
$$r = -1, -1$$
$$y_h = c_1 e^{-t} + c_2 t e^{-t}$$

$$y_p = At + B$$
$$y_p' = A$$
$$y_p'' = 0$$
$$0 + 2A + (At + B) = -t$$
$$A = -1 \qquad 2A + B = 0$$
$$B = -2A = +2$$

$$y_p = -t + 2$$

$$\boxed{y = c_1 e^{-t} + c_2 t e^{-t} - t + 2}$$

$$x = y' - 2y = (-c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} - 1) - 2c_1 e^{-t} - 2c_2 t e^{-t} + 2t - 4$$

$$\boxed{x = -3c_2 t e^{-t} + (c_2 - 3c_1) e^{-t} + 2t - 5}$$

6. (15 pts.) Find the function whose inverse Laplace transform is the solution to the initial value problem

$$y'' - y' + y = \begin{cases} t & \text{if } 0 \le t \le 3 \\ 3 & \text{if } t > 3 \end{cases}$$

$$y(0) = 1 \quad , \quad y'(0) = 2$$

$$y'' - y' + y = t\left(u(t) - u(t-3)\right) + 3u(t-3)$$

$$= tu(t) - tu(t-3) + 3u(t-3)$$

$$(s^2\mathcal{L}\{y\} - s\cdot 1 - 2) - (s\mathcal{L}\{y\} - 1) + \mathcal{L}\{y\}$$

$$= \frac{1}{s^2} - e^{-3s}\mathcal{L}\{t+3\} + e^{-3s}\mathcal{L}\{3\}$$

$$= \frac{1}{s^2} - e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right) + e^{-3s}\left(\frac{3}{s}\right)$$

$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-3s} \quad = (s^2 - s + 1)\mathcal{L}\{y\} - s - 1$$

$$\boxed{\mathcal{L}\{y\} = \frac{1}{s^2 - s + 1}\left(s + 1 + \frac{1}{s^2} - \frac{1}{s^2}e^{-3s}\right)}$$

7. (15 pts.) Find the solution to the initial value problem

$$y'' + 4y = 3 + \delta(t-4)$$
$$y(0) = 0 \quad , \quad y'(0) = 2$$

$$\mathcal{L}\{y''+4y\} = \mathcal{L}\{3+\delta(t-4)\}$$

$$s^2\mathcal{L}\{y\} - s\cdot 0 - 2 + 4\mathcal{L}\{y\} = 3\cdot\frac{1}{s} + e^{-4s}$$

$$(s^2+4)\mathcal{L}\{y\} = 2 + 3\cdot\frac{1}{s} + e^{-4s}$$

$$\mathcal{L}\{y\} = \frac{2}{s^2+4} + 3\frac{1}{s(s^2+4)} + \frac{e^{-4s}}{s^2+4}$$

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{As^2+4A+Bs^2+Cs}{s(s^2+4)}$$

$$(A+B)s^2 + Cs + 4A = 1 \longrightarrow \quad C=0 \quad 4A=1 \quad A=\frac{1}{4}$$

$$A+B=0 \quad B=-\frac{1}{4}$$

$$\mathcal{L}\{y\} = \frac{2}{s^2+4} + \frac{3}{4}\frac{1}{s} - \frac{3}{4}\frac{s}{s^2+4} + \frac{e^{-4s}}{s^2+4}$$

$$y = \mathcal{L}^{-1}\{-\} = \sin(2t) + \frac{3}{4} - \frac{1}{4}\cos(2t) + u(t-4)\frac{1}{2}\sin(t-4)$$

$$\boxed{= \frac{3}{4} + \sin(2t) - \frac{3}{4}\cos(2t) + \frac{1}{2}\sin(t-4)\,u(t-4)}$$