Math 221

Topics since the second exam

Chapter 5: Systems of Equations.

Basic idea: we have several unknown functions $x, y, \ldots$ of time t , and equations describing the derivatives of each in terms of $t, x, y, \ldots$. The goal: determine the functions $x(t), y(t), ldots$ which solve the equations *at the same time*. Initial value problem: the value of each function is specified at a specific time: $x(t_0) = x_0, y(t_0) = y_0, \ldots$.

Example: Multiple tank problem: several tanks connected by pipes, with solutions of varying concentrations in them. Applying

rate of change of amount of solute =
(rate at which solute comes in) − (rate at which solute goes out)

to each tank gives a system of equations.

A multiple tank problem gives a *linear* system of equations: each equation has the form

$$y_i' = a_1 y_1 + \cdots a_n y_n + f_i(t)$$

for some collection of constants $a_1, \ldots, a_n$ and (usually constant) function $f_i(t)$. Such systems can be solved by the *elimination method*; the first equation can be rewritten as y_n = (some expression), which can then be substituted into the remaining equations to give $n-1$ equations in $n-1$ unknown functions. The process can then be repeated, yielding, in the end, a single n-th order linear equation (with constant coefficients) in a single unknown function. This can then be solved by our earlier techniques.

[**For further details, see the handout from class!**]

Autonomous systems: The other kind of system of equations which earlier techniques can help us solve is *autonomous systems*; this means that each equation in the system has the form

$$y_i' = f_i(y_1, \ldots, y_n)$$

with no independent variable t appearing on the right-hand side. For these, we can use the *direction field* just as we did for autonomous equations before. For the sake of our exposition, we will deal with an autonomous system of two equations

$$x' = f(x, y) \quad , \quad y' = g(x, y)$$

Our solution would be a pair of functions $(x(t), y(t))$, which we can think of as describing a *parametrized curve* in the x-y plane. The tangent vector to this curve is $(x'(t), y'(t))$ = $(f(x(t), y(t)), g(x(t), y(t)) = (f(x, y), g(x, y))$. So every solution curve is tangent to the *direction field* $(f(x, y), g(x, y))$ at every point along the curve. So by drawing the direction field, we can estimate the *trajectories* of solutions (by not, in general, their actual parametrizations), by finding curves tangent to the vectors of the field.

The task of drawing a direction field can be simplified by drawing the *nullclines* of the field, that is, the curves where $f(x, y) = 0$ (vertical tangents) and where $g(x, y) = 0$ (horizontal tangents). Where such curves cross, we have *equilibrium points*. These are points where $x'(t) = y'(t) = 0$; that is, the constant x- and y-values are *constant solutions* to the system of equations.

For a linear system with constant coefficients, the basic shape of the direction field is completely determined by the roots of the auxiliary equation for the second order equation obtained from the elimination method. Complex roots give a spiral pattern around the equilibrium values (and spiralling in or out depending upon whether or not the real part of the roots are negative or not); real roots will make the direction field point towards the equilibrium or away from it, depending on if they are negative or positive; if there is one of each we have one direction pointing in and one pointing out. Details of this may be found on the handout from class!

For a more general autonomous system, we can *linearize* the equation at each equilibrium point (x_0, y_0), that is, replace our original equations with

$$x' = [f_x(x_0, y_0)]x + [f_y(x_0, y_0)]y \qquad , y' = [g_x(x_0, y_0)]x + [g_y(x_0, y_0)]y$$

The behavior of solution curves, near the equilibrium point, are well-represented by the solutions to the linearized equation.

Chapter 7: Laplace Transforms.

There is a whole different set of techniques for solving n-th order linear equations, which are based on the *Laplace transform* of a function. For a function $f(t)$, it's Laplace transform is

$$\mathcal{L}\{f\} = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The domain of $\mathcal{L}\{f\}$ is all values of s where the improper integral converges. For most basic functions f, $\mathcal{L}\{f\}$ can be computed by integrating by parts. A list of such transforms can be found on the handout from class. The most important property of the Laplace transform is that it *turns differentiation into multiplication by s*. that is:

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

more generally, for the n-th derivative:

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

The Laplace transform is a *linear operator* in the same sense that we have used the term before: for any functions f and g, and any constants a and b,

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$

(since integration is a linear operator). We can therefore use Laplace operators to solve linear (inhomogeneous) equations (with constant coefficients), by applying $\mathcal{L}$ to both sides of the equation:

$$ay'' + by' + cy = g(t)$$

becomes

$$(as^2 + bs + c)\mathcal{L}\{y\} - asy(0) - ay'(0) - by(0) = \mathcal{L}\{g\}, \text{ i.e.}$$

$$\mathcal{L}\{y\} = \frac{\mathcal{L}\{g\}(s) + asy(0) + ay'(0) + by(0)}{as^2 + bs + c}$$

So to solve our original equation, we need to find a function y whose Laplace transform is this function on the right. It turns out there is a formula (involving an integral) for the *inverse Laplace transform* $\mathcal{L}^{-1}$, which in principle will solve our problem, but the formula is too complicated to use in practice. Instead, we will develop techniques for recognizing functions as linear combinations of the functions appearing as the right-hand sides of the formulas in our Laplace transform tables. Then the function y we want is the corresponding combination of the functions on the left-hand sides of the formulas, because the Laplace transform is linear! Note that this approach incorporates the initial value data $y(0), y'(0)$ into the solution; it is naturally suited to solving initial value problems.

Our basic technique for finding solutions is *partial fractions*: we will content ourselves with a simplified form of it, sufficient for solving second order equations. The basic idea is that we need to find the inverse Laplace transform of a function having a quadratic polynomial as^2+bs+c in its denomenator. Partial fractions tells us that, if we can factor as^2+bs+c $= a(x-r_1)(x-r_2)$, where $r_1 \neq r_2$, then any function

$$\frac{ms+n}{as^2+bs+c} = \frac{A}{s-r_1} + \frac{B}{s-r_2}$$

for appropriate constants A and B. We can find the constants by writing

$$\frac{A}{s-r_1} + \frac{B}{s-r_2} = \frac{A(s-r_2)+B(s-r_1)}{(s-r_1)(s-r_2)} = \frac{Aa(s-r_2)+Ba(s-r_1)}{as^2+bs+c}$$

so we must have $ms+n = Aa(s-r_2)+Ba(s-r_1)$; setting the coefficients of the two linear functions equal to one another, we can solve for A and B. We can therefore find the inverse Laplace transform of $(ms+n)/(as^2+bs+c)$ as a combination of the inverse transforms of $(s-r_1)^{-1}$ and $(s-r_2)^{-1}$, which can be found on the tables!

If $r_1 = r_2$, then we instead write

$$\frac{ms+n}{as^2+bs+c} = \frac{A}{s-r_1} + \frac{B}{(s-r_1)^2} = \frac{a(A(s-r_1)+B)}{a(s-r_1)^2} = \frac{a(A(s-r_1)+B)}{as^2+bs+c}$$

anbd solve for A and B as before.

Finally, if we *cannot* factor as^2+bs+c (i.e, it has complex roots), we can then write it as (a times) a sum of squares, by completing the square:

$as^2+bs+c = a((s-\alpha)^2+\beta^2)$, so

$$\frac{ms+n}{as^2+bs+c} = \frac{A\beta}{a((s-\alpha)^2+\beta^2)} + \frac{B(s-\alpha)}{a((s-\alpha)^2+\beta^2)} =$$

$$\frac{A}{a}\frac{\beta}{((s-\alpha)^2+\beta^2)} + \frac{B}{a}\frac{(s-\alpha)}{((s-\alpha)^2+\beta^2)}$$

for appropriate constants A and B (which we solve for by equating the numerators), and so it is a linear combination of $\dfrac{\beta}{((s-\alpha)^2+\beta^2)}$ and $\dfrac{(s-\alpha)}{((s-\alpha)^2+\beta^2)}$, both of which appear on our tables!

Handling higher degree polynomials in the denomenator is similar; if all roots are real and distinct, we write our quotient as a linear combination of the functions $(s-r_i)^{-1}$, combine into a single fraction, and set the numerators equal; if we have repeated roots, we include terms in the sum with successively higher powers $(s-r_i)^{-k}$ (where k runs from 1 to the multiplicity of the root). Complex roots are handled by inserting the term we dealt with above into the sum.

Discontinuous external force.

One area in which Laplace transforms provide a better framework for working out solutions than our "auxiliary equation" approach is when we are trying to solve an equation

$$ay'' + by' + cy = g(t)$$

where $g(t)$ is *discontinuous*. The model for a discontinuous function is the step function $u(t)$: $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$. More generally, the function $u(t-a)$ has $u(t-a) = 1$ for $t \geq a$ and $u(t-a) = 0$ for $t < a$. So, for example, the function which is t for $3 \leq t \leq 5$, and is 0 everywhere else, can be expressed as $g(t) = t(u(t-3) - u(t-5))$.

We can find the Laplace transform of such a function by finding the transform of functions of the form $f(t)u(t-a)$, which we can do directly from the integral, by making the substitution $x = t - a$:
$\mathcal{L}\{f(t)u(t-a)\} = \int_0^\infty e^{-st} f(t)u(t-a)\ dt = \int_a^\infty e^{-st} f(t)\ dt = \int_0^\infty e^{-s(t+a)} f(t+a)\ dt = e^{-as} \int_0^\infty e^{-st} f(t+a)\ dt = e^{-as}\mathcal{L}\{f(t+a)\}$.
Turning this around, we find that the inverse Laplace transform of the function $e^{-as}\mathcal{L}\{f\}(s)$ is $f(t-a)u(t-a)$. So if we can find the inverse transform of a function $F(s)$ (in our tables), this tells us how to find the inverse transform of $e^{-as}F(s)$. This is turn gives us a method for solving any initial value problem, in principle, whose inhomogeneous term $f(t)$ has finitely many values where it is discontinuous, by writing it as a sum of functions of the form $f_i(t)u(t-a_i)$.

For example, to find the solution to the differential equation

$y'' + 2y' + 5y = g(t)$, $y(0) = 2$, $y'(0) = 1$, where $g(t)$ is the function which is 5 for $2 \leq t \leq 4$ and 0 otherwise, we would (after taking Laplace transforms and simplifying) need to find the inverse Laplace transform of the function

$$F(s) = \frac{2s+5}{s^2+2s+5} + \frac{5(e^{-2s} - e^{-4s})}{s(s^2+2s+5)}$$

Applying our partial fractions techniques, we find that

$$F(s) = 2\frac{s+1}{(s+1)^2+2^2} + \frac{3}{2}\frac{2}{(s+1)^2+2^2} + \left(\frac{1}{s} - \frac{s+1}{(s+1)^2+2^2}\frac{1}{2} - \frac{2}{(s+1)^2+2^2}\right)e^{-2s}$$
$$-\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+2^2} - \frac{1}{2}\frac{2}{(s+1)^2+2^2}\right)e^{-4s}$$

We can apply $\mathcal{L}^{-1}$ to each term, using $\mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f\}(s)\} = f(t-a)u(t-a)$ for the last 6 terms (since after removing e^{-2s} and e^{-4s} the remainder of each term is in our tables). For example,

$\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+2^2}e^{-2s}\} = e^{-(t-2)}\cos(2(t-2))u(t-2)$. The final solution, as the interested reader can work out, is

$$y = 2e^{-t}\cos(2t) + \frac{3}{2}e^{-t}\sin(2t) + [1 - e^{-(t-2)}\cos(2(t-2)) - \frac{1}{2}e^{-(t-2)}\sin(2(t-2))]u(t-2)$$
$$-[1 - e^{-(t-4)}\cos(2(t-4)) - \frac{1}{2}e^{-(t-4)}\sin(2(t-4))]u(t-4)$$