Solutions

Name:

Math 221 Section 3

Final Exam

Show all work. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (15 pts.) Find the general solution to the differential equation

$$(x^2+1)y' + 2xy = x^2$$

$$p(x) = \frac{2x}{x^2+1} \qquad g(x) = \frac{x^2}{x^2+1}$$

$$y' + \frac{2x}{x^2+1}y = \frac{x^2}{x^2+1}$$

$$e^{\int p(x)dx} = e^{\int \frac{2x}{x^2+1}dx} = e^{\int \frac{du}{u}\big|_{u=x^2+1}} = e^{\ln(x^2+1)} = x^2+1$$

$$y = e^{-\int p(x)dx}\int e^{\int p(x)dx} g(x)\,dx$$

$$= \frac{1}{x^2+1}\int (x^2+1)\left(\frac{x^2}{x^2+1}\right)dx$$

$$= \frac{1}{x^2+1}\int x^2\,dx = \frac{1}{x^2+1}\left(\frac{1}{3}x^3 + c\right)$$

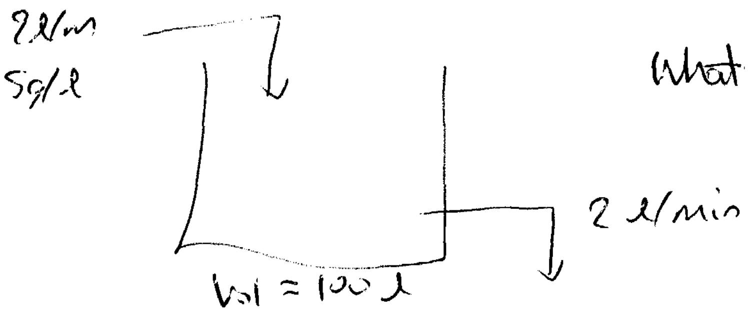
or

$$x^2 = (x^2+1)y' + (2x)y = ((x^2+1)y)'$$

$$(x^2+1)y = \int x^2\,dx = \frac{1}{3}x^3 + C$$

$$y = \frac{\frac{1}{3}x^3 + C}{x^2+1}$$

2. (15 pts.) A tank which initially holds 100 liters of water with 200g of salt in solution is draining out of the tank at a rate of 2 liters per minute. At the same time, solution with a concentration of 5 g per liter is being poured in at a rate of 2 liters per minute. What will the concentration of the well-stirred solution be after 10 minutes?



What is $x(10)$?

$x(0) = 200$

$$x' = 5 \cdot 2 - \frac{x}{100} \cdot 2$$

$$x' + \frac{2}{100} x = 10 \qquad x(0) = 200 \qquad = e^{\frac{2}{100} t}$$

$$x(t) = e^{-\int \frac{2}{100} dt} \left(e^{\int \frac{2}{100} dt} \cdot 10 \, dt \right)$$

$$= e^{-\frac{2}{100} t} \left(10 \cdot \frac{100}{2} e^{\frac{2}{100} t} + C \right)$$

$$= 500 + C e^{-\frac{2}{100} t}$$

$$x(0) = 200 = 500 + C \qquad C = -300$$

$$x(t) = 500 - 300 e^{-\frac{2t}{100}}$$

$$\boxed{x(100) = 500 - 300 e^{-\frac{20}{100}}}$$

$$\text{concentration} = \frac{500 - 300 e^{-\frac{20}{100}}}{100}$$

3. (15 pts.) Find the solution to the initial value problem

$$x^2y'' + 2xy' - 6y = 0$$
$$y(1) = 5 \quad , \quad y'(1) = 5$$

Cauchy - Euler!

$r(r-1) + 2r - 6 = 0$

$r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$y = c_1 x^{-3} + c_2 x^2$ $\qquad$ $y' = -3c_1 x^{-4} + 2c_2 x$

$y(1) = c_1 + c_2 = 5$ $\qquad$ $3c_1 + 3c_2 = 15$

$y'(1) = -3c_1 + 2c_2 = 5$ $\qquad$ add

$5c_2 = 20$, $c_2 = 4$

$c_1 = 5 - c_2 = 5 - 4 = 1$

$$\boxed{y = x^{-3} + 4x^2}$$

4. (15 pts.) Use variation of parameters to find a particular solution to the second order equation

$$y'' - 2y' + y = e^x \ln x$$

homogeneous solutions:

$$y''-2y'+y=0$$
$$r^2-2r+1=0$$
$$(r-1)^2=0$$
$$r=1,1$$

$$y_1=e^x$$
$$y_2=xe^x$$

$$W=\begin{vmatrix} e^x & xe^x \\ e^x & xe^x+e^x \end{vmatrix} = xe^{2x}+e^{2x}-xe^{2x} = e^{2x}$$

$$y=c_1y_1+c_2y_2$$

$$c_1=\int \frac{\begin{vmatrix} 0 & xe^x \\ e^x\ln x & xe^x+e^x \end{vmatrix}}{e^{2x}}\,dx = \int \frac{-(x\ln x)e^{2x}}{e^{2x}}\,dx = \int \overset{x^2/2}{x}\,\underset{1/x}{\ln x}\,dx$$

$$= -\frac{x^2}{2}\ln x + \int \frac{x}{2}\,dx = -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

$$c_2=\int \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x\ln x \end{vmatrix}}{e^{2x}}\,dx = \int \frac{e^{2x}\ln x}{e^{2x}}\,dx = \int \overset{x}{1}\cdot\underset{1/x}{\ln x}\,dx$$

$$= x\ln x - \int 1\,dx = x\ln x - x$$

$$y=\left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right)e^x + (x\ln x - x)xe^x$$

$$= \cancel{\tfrac{3}{2}x^2\ln x - \tfrac{5}{4}x^2}\quad \boxed{= \frac{1}{2}x^2e^x\ln x - \frac{3}{4}x^2e^x}$$

5. (10 pts.) Find the solution to the initial value problem

$$x' = 3x + y \quad , \quad y' = 5x - y$$
$$x(0) = 5 \quad , \quad y(0) = -7$$

$y = x' - 3x$

$y' = x'' - 3x'$

$x'' - 3x' = 5x - (x' - 3x) = 8x - x'$

$x'' - 2x' - 8x = 0$

$r^2 - 2r - 8 = 0$

$(r-4)(r+2) = 0$

$x(t) = c_1 e^{4t} + c_2 e^{-2t}$

$x'(t) = 4c_1 e^{4t} - 2c_2 e^{-2t}$

$x(0) = 5$

$x'(0) = 3x(0) + y(0)$
$= 3 \cdot 5 - 7$
$= 8$

$x(0) = c_1 + c_2 = 5$

$x'(0) = 4c_1 - 2c_2 = 8$

$2c_1 + 2c_2 = 10$

$6c_1 = 18 \qquad c_1 = 3$

$c_2 = 5 - c_1 = 2$

$x(t) = 3e^{4t} + 2e^{-2t}$

$y(t) = x'(t) - 3x(t) =$

$= (12 - 9)e^{4t} + (-4 - 6)e^{-2t}$

$= 3e^{4t} - 10e^{-2t}$

$$\boxed{x(t) = 3e^{4t} + 2e^{-2t} \ , \ y(t) = 3e^{4t} - 10e^{-2t}}$$

6. (15 pts.) Sketch the direction field for the system of equations

$$x' = x + y \quad , \quad y' = 3y - xy$$

and sketch the trajectory of the solution passing through the point (1,2) .

(find the equilibrium solutions),

$x' = 0 = x + y$, $y = -x$

$y' = 0 = 3y - xy = y(3-x)$ → $y = 0$ or $x = 3$

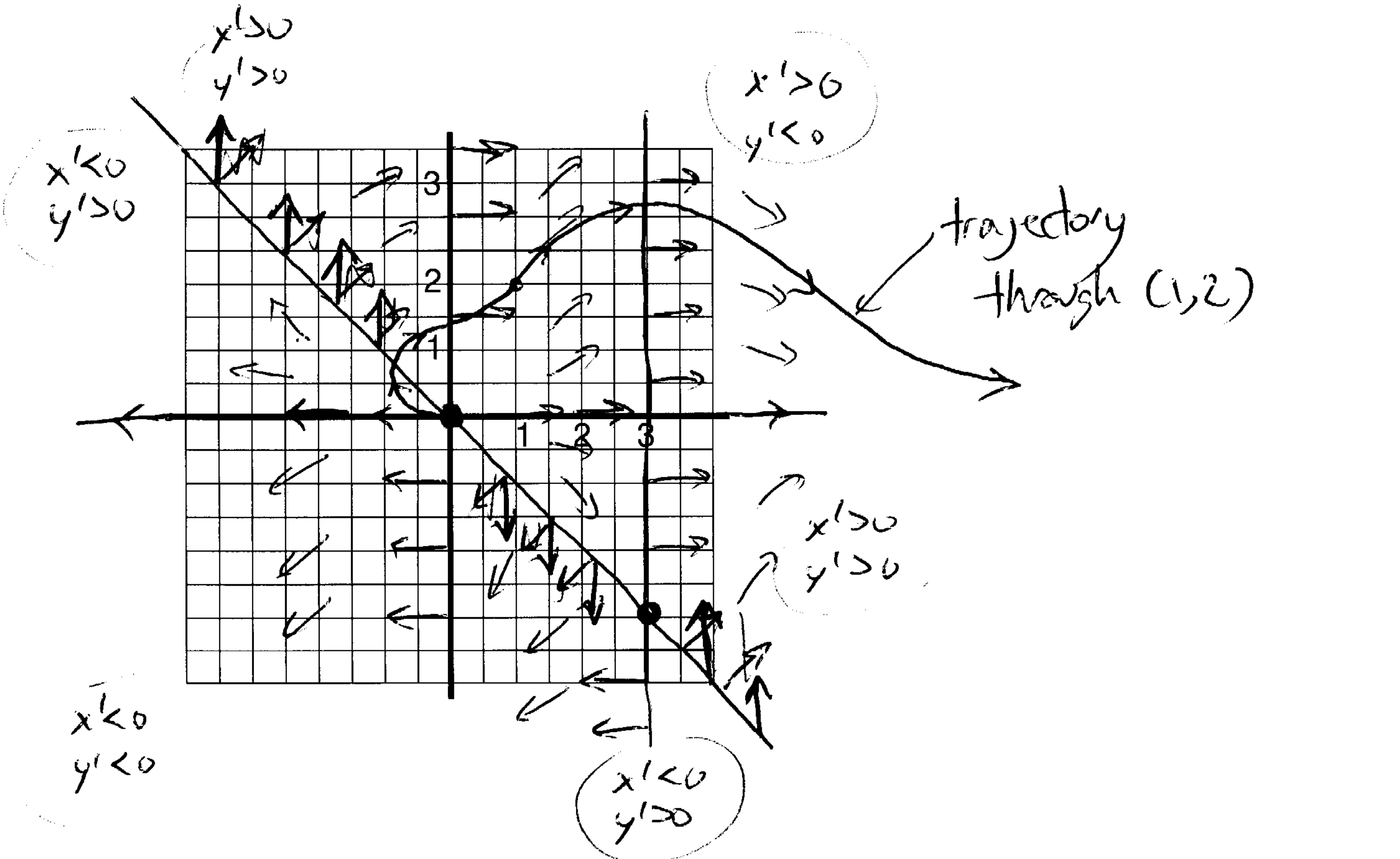
$x' = 0$ and $y' = 0$: $y = 0 \Rightarrow x = -y = 0$ $(0,0)$

$x = 3 \Rightarrow y = -x = -3$ $(3,-3)$ ⇐ equilibrium solutions

$x' = 0$ on line $y = -x$

$y' = 0$ on lines $y = 0$; $x = 3$

Direction of vectors is same in each region b/w these lines



7. (15 pts.) Find the function $F(s)$ whose inverse Laplace transform gives a solution to the initial value problem

$$y'' + 3y' - 7y = \begin{cases} 2 & \text{if } 1 \le t < 3 \\ t^2 & \text{if } 3 \le t \le 6 \\ 0 & \text{otherwise} \end{cases} \quad = f(t)$$

$$y(0) = 3 \quad , \quad y'(0) = -1$$

$$\mathcal{L}\{y'' + 3y' - 7y\} = s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + 3s\mathcal{L}\{y\} - 3y(0) + 7\mathcal{L}\{y\}$$

$$= (s^2+3s-7)\mathcal{L}\{y\} - 3s + 1 - 9$$

$$= (s^2+3s-7)\mathcal{L}\{y\} - 3s - 8$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\, 2(u(t-1) - u(t-3)) + t^2(u(t-3) - u(t-6))\}$$

$$= 2\mathcal{L}\{u(t-1)\} - 2\mathcal{L}\{u(t-3)\} + \mathcal{L}\{t^2u(t-3)\} - \mathcal{L}\{t^2u(t-6)\}$$

$$= 2\frac{e^{-s}}{s} - 2\frac{e^{-3s}}{s} + e^{-3s}\mathcal{L}\{(t+3)^2\} - e^{-6s}\mathcal{L}\{(t+6)^2\}$$

$$= 2\frac{e^{-s}}{s} - 2\frac{e^{-3s}}{s} + e^{-3s}\left(\frac{2}{s^3} + 6\frac{1}{s^2} + 9\frac{1}{s}\right) - e^{-6s}\left(\frac{2}{s^3} + 12\frac{1}{s^2} + 36\frac{1}{s}\right)$$

$$(s^2+3s-7)\mathcal{L}\{y\} = 3s + 8 + \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{y\} = F(s) = \boxed{\frac{1}{s^2+3s-7}\left(3s + 8 + \frac{2e^{-s}}{s} - \frac{2e^{-3s}}{s} + e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right) - e^{-6s}\left(\frac{2}{s^3} + \frac{12}{s^2} + \frac{36}{s}\right)\right)}$$