**Math 1650**

**Topics for second exam**

(Technically, everything covered on the <u>first</u> exam, <u>plus</u>...)

**Chapter 2:** Polynomials

§3: Polynomial division

root $a$ of $f \leftrightarrow$ factor $(x-a)$ of $f(x)$

reason: polynomial (long) division

$f(x) = (x-a)g(x) + b$ ; a=root, then b=0

polonomial = (divisor)(quotient) + remainder

degree of remainder is less than degree of divisor

synthetic division: fast method to divide by (x-a)

§4: Real zeros of polynomial functions

$f(x) = a_n x^n + \cdots + a_1 x + a_0$

'Counting' zeros of $f$

Descartes' rule of signs

p=number of positive roots of $f$, q=number of negative roots of $f$

(number of changes in sign of coeffs of $f$) $-$ p is $\geq$0 and <u>even</u>

(number of changes in sign of coeffs of $f(-x)$) $-$ q is $\geq$0 and <u>even</u>

Rational roots test

If $a_n, \dots, a_0$ are all integers, $a_n \neq 0$, and $r = p/q$ is a <u>rational</u> root of $f$, then $q$ divides $a_n$ evenly and $p$ divides $a_0$ evenly.

backwards: can show roots of a polynomial <u>can't</u> be rational.

Bounding roots: start with $a_n > 0$.

If $c > 0$ and the bottom row after synthetic division of $f$ using $c$ are all $\geq$0, then no root of $f$ is bigger than $c$.

If $c$ <0 and the bottom alternates sign, then no root of $f$ is smaller than $c$.

§5: Complex numbers

Some polynomials have no roots, e.g., $f(x) = x^2 + 1$ . Invent some!

$i = \sqrt{-1}$, pretend $i$ behaves like a real number

complex numbers: standard form $z = a + bi$ ; addition, subtraction, multiplication

division: complex conjugate $\overline{z} = a - bi$

$z \cdot \overline{z} = a^2 + b^2$ (<u>real</u>!) ; $z_1/z_2 = (z_1 \cdot \overline{z_2})/(z_2 \cdot \overline{z_2})$

$a, b > 0$, then $\sqrt{-a} \cdot \sqrt{-b} = -\sqrt{(-a)(-b)}$ (unfortunately)

§6: The fundamental theorem of algebra

FTA: Every polynomial $f(x)$ (with coefficients in $\mathbf{C}$ or $\mathbf{R}$) has a complex root $r$ ; $f(r)$=0

Every polynomial factors into linear factors (with coefficients in $\mathbf{C}$)

FTA says it can be done; it doesn't tell you <u>how</u> to do it!

Conjugate pairs; if coeffs of $f$ are <u>real</u>, and $r$ is a root, then so is $\overline{r}$

$(x-r)(x-\overline{r})$ has <u>real</u> coeffs

every polynomial with real coeffs factors in linear and irreducible quadratic factors.

§7: Rational functions

rational function = quotient of polynomials

$p(x) = a_n x^n + \cdots + a_0$, $q(x) = b_m x^m + \cdots + b_0$ ; $f(x) = p(x)/q(x)$

domain = where $q(x) \neq 0$

vertical asymptote $x = a$ : $f(x) \to \pm\infty$ as $x \to a$

horizontal asymptote: $f(x) \to a$ as $x \to \pm\infty$

$n < m$ : horiz. asymp. $y = 0$

$n = m$ : horiz. asymp. $y = a_n/b_m$

$n > m$ : no horiz. asymp.

Slant asymptote: $n = m + 1$ . Asymp. = linear part from division of $p(x)$ by $q(x)$

**Chapter 3:** Exponential and logarithmic functions

§1: Exponential functions

exponential expressions $a^b$

Rules: $a^{b+c} = a^b a^c$ ; $a^{bc} = (a^b)^c$ ; $(ab)^c = a^c b^c$

Function $f(x) = a^x$ ; approximate $f(x)$ by $f$(rational number close to $x$)

Domain: $\mathbf{R}$ ; range: $(0, \infty)$ ; horiz. asymp. $y = 0$

Graphs:

$a > 1$ $\qquad\qquad$ $0 < a < 1$

Most natural base: $e = 2.718281829459045.....$

Exponential growth: compound interest

$P$=initial amount, $r$=interest rate, compounded $n$ times/year

$A(t) = P \cdot (1 + r/n)^{nt}$

$n \to \infty$, continuous compounding : $A(t) = Pe^{rt}$

Radioactive decay: half-life = k ($A(k) = A(0)/2$)

$A(t) = A(0)(1/2)^{t/k}$

§2: Logarithmic functions

$\log_a x$ = the number you raise $a$ to to get $x$

$\log_a x$ is the <u>inverse</u> of $a^x$

$a$ = base of the logarithm

$\log_a(a^x) = x$, all $x$ ; $a^{\log_a x} = x$, all $x > 0$

Domain: all $x > 0$ ; range: all $x$

Graph = reflection of graph of $a^x$ across line $y = x$

vertical asymptote: $x = 0$

natural logarithm: $\log_e x = \ln x$

§3: Properties of logarithms

logarithms undo exponentials; properties are 'reverse' of exponentials

$\log_a(bc) = \log_a b + \log_a c$ ; $\log_a(b^c) = c \log_a b$

$(\log_b c)(\log_a b) = \log_a(b^{\log_b c}) = \log_a c$; so $\log_b c = \dfrac{\log_a c}{\log_a b}$

E.g., $a = e$ : $\log_b c = \dfrac{\ln c}{\ln b}$

§4: Exponential and logarithmic equations

exponential equation: take logs!

$a^{blah} = bleh$, then $(blah)\ln a = \ln(bleh)$

$(2^x - 3)(2^x - 7) = 0$, then $2^x$=3 <u>or</u> $2^x$=7

logarithmic equation: combine into a single log (one on each side?) and exponentiate both sides

Application: doubling time
time for investment to triple at interest rate of $r$ compounded $n$ times/year:
solve $(1 + r/n)^{nt} = 3$