Math 221 - Section 5 - Fall 2009

A quick guide to sketching direction fields

Section 1.3 of the text discusses approximating solutions of differential equations using graphical methods, via direction (i.e., slope) fields. However, there is one idea, not mentioned in the book, that is very useful to sketching and analyzing direction fields, namely *nullclines and isoclines.* The standard form of a first order DE for which there is a slope field is:

$$\frac{dy}{dx} = f(x, y)$$

To sketch the direction field of such a system, at each point (x_0, y_0) in the xy-plane, we draw a vector starting at (x_0, y_0) with slope $f(x_0, y_0)$.

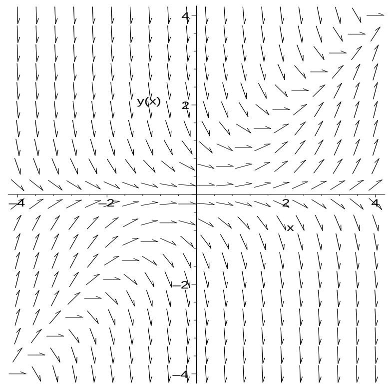
Definition of nullcline. The **nullcline** is the set of points in the direction field so that $\frac{dy}{dx} = 0$. Geometrically, these are the points where the vectors are horizontal. Algebraically, we find the nullcline by solving $f(x, y) = 0$.

How to use the nullcline. Consider the DE

$$\frac{dy}{dx} = y(x - y).$$

To find the nullcline, we solve the equation $y(x-y) = 0$. This gives two lines, the line $y = x$ and the x-axis.

Vectors in the direction field point upward if $y(x - y) > 0$, which occurs in two cases: when both $y > 0$ and $x - y > 0$ (above the x axis and below the line $y = x$), and when both $y < 0$ and $x - y < 0$ (below the x axis and above the line $y = x$). Similarly, vectors point downward if $y(x - y) < 0$, which occurs when both $y > 0$ and $x - y < 0$, and when both $y < 0$ and $x - y > 0$.



We can use this direction field to analyze how the solution $y(x)$ of the DE $y' = y(x - y)$ will behave as $x \to \infty$, depending on the initial condition $y(x_0) = y_0$. If $x_0 > 0$, then as $x \to \infty$, the solution $y(x) \to \infty$. If $x_0 = 0$, then $y(x) \to 0$, and if $x_0 < 0$, then $y(x) \to -\infty$.

Definition of isocline. An **isocline** is a set of points in the direction field for which there is a constant c with $\frac{dy}{dx} = c$ at these points. Geometrically, the direction field arrows at the points of the isocline all have the same slope. Algebraically, we find the isocline for a constant c by solving $f(x, y) = c$.

How to use the method of isoclines.

Step 1: Find the nullcline and draw in the corresponding horizontal arrows.

Step 2: Find the regions of the plane in which vectors point upward or downward, as described above.

Step 3: Chose several constants, both positive and negative, find their isoclines, and plot the corresponding vector arrows.

Consider the DE

$$\frac{dy}{dx} = x^2 + y^2 - 1.$$

The nullcline is the solution of the equation $x^2 + y^2 = 1$, which is the circle centered at (0,0) with radius 1. The points on this cirle have horizontal arrows.

The direction field arrows will point upward when $x^2+y^2-1 > 0$, that is when $x^2+y^2 > 1$, outside of the nullcline circle. The arrows will point downward when $x^2+y^2-1 < 0$, which is the set of points inside of the nullcline circle.

The isocline corresponding to the constant 3 is the set of points (x, y) for which $x^2+y^2-1 = 3$. This is the circle $x^2 + y^2 = 4$ centered at (0,0) of radius 2. The points on this circle have arrows of slope 3.

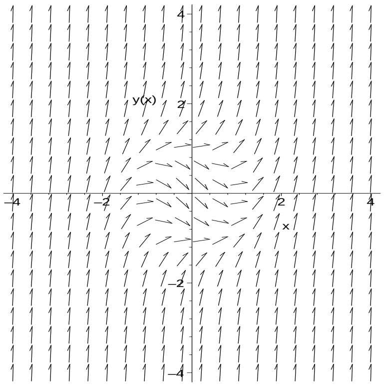
Similarly, the isoclines corresponding to the constants 8 and 15 are the circles centered at (0,0) of radius 3 and 4, respectively. Thus the points on the circle of radius 3 have arrows of slope 8, and the points on the circle of radius 4 have arrows of slope 15.

The isocline corresponding to the constant -1 is the set of points (x, y) for which $x^2 + y^2 - 1 = -1$, that is, $x^2 + y^2 = 0$. This is the single point (0,0), at which the direction field vector has slope -1.

The isoclines corresponding to any constant $c < -1$ contain no points at all.

The isocline corresponding to the constant $-\frac{3}{4}$ is the solution of the equation $x^2+y^2-1 = -\frac{3}{4}$ or $x^2 + y^2 = \frac{1}{4}$, which is the circle centered at (0,0) of radius $\frac{1}{2}$. The points on this circle have direction field arrows of slope $-\frac{3}{4}$.

Similarly, the isocline corresponding to $-\frac{8}{9}$ is the circle centered at (0,0) of radius $\frac{1}{3}$, at which the arrows have slope $-\frac{8}{9}$.



In this example, for every possible initial condition $y(x_0) = y_0$, we have that as $x \to \infty$, then $y(x) \to \infty$.

Exercises. Using the method of isoclines, sketch the direction fields, and discuss the possible fates of solutions, depending on initial conditions, for the following DEs.

(1.3.21)	$\frac{dy}{dx} = x + y.$	(1.3.24)	$\frac{dy}{dx} = x + \frac{1}{2}y^2.$
(1.3.25)	$\frac{dv}{dt} = 32 - 1.6v.$	(1.3.26)	$\frac{dP}{dt} = 0.0225P - 0.0003P^2.$
(1.3.28)	$x\frac{dy}{dx} = y.$	(6)	$\frac{dy}{dx} = x(10 - x - y).$
(2.2.13)	$\frac{dx}{dt} = (x + 2)(x - 2)^2.$	(2.2.16)	$\frac{dx}{dt} = x^2(x^2 - 4).$