Math 221, Section 5

Quiz number 7 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Use variation of parameters to find a particular solution to the differential equation

$$y'' - 2y' + y = \frac{e^x}{x^2+1} = f(x) \ .$$

A set of fundamental solutions to the associated homogeneous equation

$$y'' - 2y' + y = 0$$

can be obtained from the characteristic equation

$$a^2 - 2a + 1 = (a-1)^2 = 0 \ ,$$

with roots $a = 1$ and $a = 1$. So the fundamental solutions are $y_1 = e^x$ and $y_2 = xe^x$. Their Wronskian is

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = (e^x)(xe^x + e^x) - (e^x)(xe^x) = (e^x)^2 \ .$$

So by variation of parameters, a particular solution to our DE is

$$y_p = c_1 e^x + c_2 xe^x, \text{ where}$$

$$c_1 = \int \frac{-y_2 f(x)}{W(y_1, y_2)} \ dx = \int \frac{-xe^x}{(e^x)^2} \frac{e^x}{x^2+1} \ dx = \int \frac{-x}{x^2+1} \ dx$$

Making the substitution $u = x^2 + 1$, $du = 2x \ dx$, we have

$$c_1 = \int -\frac{1}{2}\frac{du}{u}\bigg|_{u=x^2+1} = -\frac{1}{2}\ln u\bigg|_{u=x^2+1} = -\frac{1}{2}\ln(x^2+1) \ . \text{ We also have}$$

$$c_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)} \ dx = \int \frac{e^x}{(e^x)^2} \frac{e^x}{x^2+1} \ dx = \int \frac{dx}{x^2+1} = \arctan(x) \ .$$

So

$$y_p = -\frac{1}{2} e^x \ln(x^2+1) + xe^x \arctan(x)$$

is a particular solution to our differential equation.